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Total Dominator Colorings in Cycles

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Abstract: Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in cycles.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely k-dominator coloring, Smarandachely k-dominator chromatic number.

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§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [3]. Let $G = (V, E)$ be a graph of order n with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\underset{v \in S}{\cup} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S$.

A subset S of V is called a total dominating set if every vertex in V is adjacent to some vertex in S. A total dominating set is minimal total dominating set if no proper subset of S is a total dominating set of G. The total domination number γ_t is the minimum cardinality taken over all minimal total dominating sets of G. A γ_t-set is any minimal total dominating set with cardinality γ_t.

A proper coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \ldots, u_p\}$ and $Ç = \{C_1, C_2, C_3, \ldots, C_n\}, n \leqslant p$ be a collection of subsets $C_i \subset V$. A color represented in a vertex u is called a non-repeated color if there exists one color class $C_i \in Ç$ such that $C_i = \{u\}$.

Let G be a graph without isolated vertices. For an integer $k \geqslant 1$, a Smarandachely k-dominator coloring of G is a proper coloring of G with the extra property that every vertex

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in G properly dominates a k-color classes and the smallest number of colors for which there exists a Smarandachely k-dominator coloring of G is called the Smarandachely k-dominator chromatic number of G and is denoted by $\chi_{td}^S(G)$. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper, we determine total dominator chromatic number in cycles.

Throughout this paper, we use the following notations.

Notation 1.1 *Usually, the vertices of C_n are denoted by $u_1, u_2, \ldots, u_n$ in order. For $i < j$, we use the notation $\langle [i,j] \rangle$ for the subpath induced by $\{u_i, u_{i+1}, \ldots, u_j\}$. For a given coloring C of C_n, $C|\langle [i,j] \rangle$ refers to the coloring C restricted to $\langle [i,j] \rangle$.*

We have the following theorem from [1].

Theorem 1.2([1]) *Let G be any graph with $\delta(G) \geq 1$. Then*

$$\max\{\chi(G), \gamma_t(G)\} \leq \chi_{td}(G) \leq \chi(G) + \gamma_t(G).$$

Definition 1.3 *We know from Theorem (1.2) that $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n)+1, \gamma_t(P_n)+2\}$. We call the integer n, good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1$, $\chi_{td}(P_n) = \gamma_t(P_n)$).*

First, we prove a result which shows that for large values of n, the behavior of $\chi_{td}(P_n)$ depends only on the residue class of $n mod 4$ [More precisely, if n is good, $m > n$ and $m \equiv n (\bmod 4)$ then m is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

Fact 1.4 Let $1 < i < n$ and let C be a td-coloring of P_n. Then, if either u_i has a repeated color or u_{i+2} has a non-repeated color, $C|\langle [i+1,n] \rangle$ is also a td-coloring. This fact is used extensively in this paper.

§2. Determination of $\chi_{td}(C_n)$

It is trivially true that $\chi_{td}(C_3) = 3$ and $\chi_{td}(C_4) = 2$. We assume $n \geq 5$.

Lemma 2.1 *If P_n has a minimum td-coloring in which the end vertices have different colors, then $\chi_{td}(C_n) \leq \chi_{td}(P_n)$.*

Proof Join $u_1 u_n$ by an edge and we get an induced td-coloring of C_n. □

Corollary 2.2 $\chi_{td}(C_n) \leq \chi_{td}(P_n)$ *for* $\forall n \neq 3, 11, 18$.

Lemma 2.3 *If C_n has a minimal td-coloring in which either there exists a color class of the form $N(x)$, where x is a non-repeated color or no color class of the form $N(x)$, then*

$\chi_{td}(P_n) \leq \chi_{td}(C_n)$.

Proof We have assumed $n > 3$. If $n = 3$, conclusion is trivially true. We have the following two cases.

Case 1 C_n has a minimal td-coloring C in which there is a color class of the form $N(x)$, where x is a non-repeated color. Let C_n be the cycle $u_1u_2\ldots u_nu_1$. Let us assume $x = u_2$ has a non-repeated color n_1 and $N(x) = \{u_1, u_3\}$ is the color class of color r_1. Then u_{n-1} has a non-repeated color since u_n has to dominate a color class which must be contained in $N(u_n) = \{u_1, u_{n-1}\}$. Thus $C|\langle[1,n]\rangle$ is a td-coloring. Thus $\chi_{td}(P_n) \leq \chi_{td}(C_n)$.

Case 2 There exists C_n has a minimal td-coloring which has no color class of the form $N(x)$. It is clear from the assumption that any vertex with a non-repeated color has an adjacent vertex with non-repeated color. We consider two sub cases.

Subcase *a* There are two adjacent vertices u, v with repeated color. Then the two vertices on either side of u, v say u_1 and v_1 must have non-repeated colors. Then the removal of the edge uv leaves a path P_n and $C|\langle[1,n]\rangle$ is a td-coloring.

Subcase *b* There are adjacent vertices u, v with u (respectively v) having repeated (respectively non-repeated) color. Then consider the vertex $u_1(\neq v)$ adjacent to u. We may assume u_1 has non-repeated color (because of sub case (a)). v_1 must also have a non-repeated color since v must dominate a color class and u has a repeated color. Once again, $C|(C_n - uv)$ is a td-coloring and the proof is as in sub case (a). Since either sub case (a) or sub case (b) must hold, the lemma follows. □

Lemma 2.4 $\chi_{td}(C_n) = \chi_{td}(P_n)$ *for* $n = 8, 13, 15, 22$.

proof We prove for $n = 22$. By Lemma 2.1, $\chi_{td}(P_{22}) \geq \chi_{td}(C_{22})$. Let $\chi_{td}(C_{22}) < \chi_{td}(P_{22}) = 14$. Then by Lemma 2.3, C_{22} has a minimal td-coloring in which there is a color class of the form $N(x)$, where x is a repeated color (say C_1). Suppose $x = u_2$ First, we assume that the color class of u_2 is not $N(u_1)$ or $N(u_3)$. Then we have u_4, u_5, u_{22}, u_{21} must be non-repeated colors.

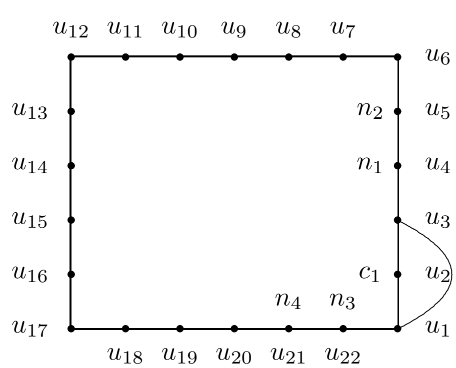

Fig.1

Then $C|\langle[6,20]\rangle$ is a coloring (which may not be a td-coloring for the section) with 8 colors including $C_1 \Rightarrow$ The vertices u_7 and u_{19} have the color C_1. (The sets $\{u_6,u_8\},\{u_7,u_9\},\{u_{10},u_{12}\}$, $\{u_{11},u_{13}\},\{u_{14},u_{16}\},\{u_{15},u_{17}\},\{u_{18},u_{20}\}$ must contain color classes. Therefore the remaining vertex u_{19} must have color C_1. Similarly, going the other way, we get u_7 must have color C_1). Then $\{u_6,u_8\},\{u_{18},u_{20}\}$ are color classes and u_9,u_{10},u_{16},u_{17} are non-repeated colors. This leads $\langle[11,15]\rangle$ to be colored with 2 colors including C_1, which is not possible. Hence $\chi_{td}(C_{22}) = 14 = \chi_{td}(P_{22})$. If the color class of u_2 is $N(u_1)$ or $N(u_3)$, the argument is similar. Proof is similar for n = 8,13,15. □

Lemma 2.5 *Let n be a good integer. Then $\chi_{td}(P_n) \le \chi_{td}(C_n)$*

Proof We use induction on n. Let $u_1,u_2,\ldots,u_n$ be vertices of C_n in order. Let C be a minimal td-coloring of C_n. For the least good integers in their respective residue classes mod 4 is $8,13,15,22$, the result is proved in the previous Lemma 2.4. So we may assume that the result holds for all good integers $< n$ and that $n-4$ is also a good integer. First suppose, there exists a color class of the form $N(x)$. Let $x = u_2$. Suppose u_2 has a repeated color. Then we have u_4,u_5,u_n,u_{n-1} must be non-repeated color. We remove the vertices $\{u_1,u_2,u_3,u_n\}$ and add an edge u_4u_{n-1} in C_n. Therefore,we have the coloring $C|\langle[4,n-1]\rangle$ is a td-coloring with colors $\chi_{td}(C_n)-2$. Therefore, $\chi_{td}(C_n) \ge 2+\chi_{td}(C_{n-4}) \ge 2+\chi_{td}(P_{n-4}) = \chi_{td}(P_n)$.

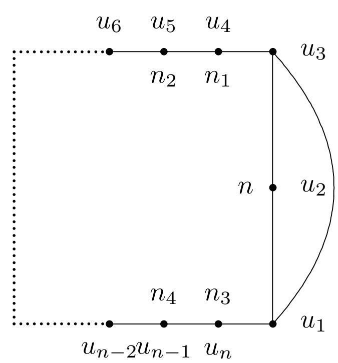

Fig.2

If x is a non-repeated color, then by Lemma 2.3, $\chi_{td}(P_n) \le \chi_{td}(C_n)$. If there is no color class of the form $N(x)$, then $\chi_{td}(P_n) \le \chi_{td}(C_n)$. □

Theorem 2.6 $\chi_{td}(C_n) = \chi_{td}(P_n)$, *for all good integers n.*

Proof The result follows from Corollary 2.2 and Lemmas 2.4 and 2.5. □

Remark Thus the $\chi_{td}(C_n) = \chi_{td}(P_n)$ for $n = 8,12,13,15,16,17$ and $\forall n \ge 19$. It can be verified that $\chi_{td}(C_n) = \chi_{td}(P_n)$ for $n = 5,6,7,9,10,14$ and that $\chi_{td}(C_n) = \chi_{td}(P_n)+1$ for $n = 3,11,18$ and that $\chi_{td}(P_4) = \chi_{td}(C_4)+1$.

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