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On the Smarandache totient function and the Smarandache power sequence

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Abstract For any positive integer n, let $SP(n)$ denotes the Smarandache power sequence. And for any Smarandache sequence $a(n)$, the Smarandache totient function $St(n)$ is defined as $\varphi(a(n))$, where $\varphi(n)$ is the Euler totient function. The main purpose of this paper is using the elementary and analytic method to study the convergence of the function $\frac{S_1}{S_2}$, where $S_1 = \sum_{k=1}^{n}\left(\frac{1}{St(k)}\right)^2$, $S_2 = \left(\sum_{k=1}^{n}\frac{1}{St(k)}\right)^2$, and give an interesting limit Theorem.

Keywords Smarandache power function, Smarandache totient function, convergence.

§1. Introduction and results

For any positive integer n, the Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that $n \mid m^m$, where m and n have the same prime divisors. That is,

$$SP(n) = \min\left\{m:\ n \mid m^m,\ m \in N,\ \prod_{p|n} p = \prod_{p|m} p\right\}.$$

For example, the first few values of $SP(n)$ are: $SP(1) = 1$, $SP(2) = 2$, $SP(3) = 3$, $SP(4) = 2$, $SP(5) = 5$, $SP(6) = 6$, $SP(7) = 7$, $SP(8) = 4$, $SP(9) = 3$, $SP(10) = 10$, $SP(11) = 11$, $SP(12) = 6$, $SP(13) = 13$, $SP(14) = 14$, $SP(15) = 15$, $\cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of $SP(n)$. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(8) = 4$, $SP(3) = 3$, $SP(24) = 6 \neq SP(3) \times SP(8)$. But for most n, we have $SP(n) = \prod_{p|n} p$, where $\prod_{p|n}$ denotes the product over all different prime divisors of n. If $n = p^\alpha$, $k \cdot p^k + 1 \leq \alpha \leq (k+1)p^{k+1}$, then we have $SP(n) = p^{k+1}$, where $0 \leq k \leq \alpha - 1$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, for all α_i $(i = 1, 2, \cdots, r)$, if $\alpha_i \leq p_i$, then $SP(n) = \prod_{p|n} p$.

About other properties of the function $SP(n)$, many authors had studied it, and gave some interesting conclusions. For example, in reference [4], Zhefeng Xu had studied the mean value properties of $SP(n)$, and obtained a sharper asymptotic formula:

$$\sum_{n\leq x} SP(n) = \frac{1}{2}x^2 \prod_{p}\left(1 - \frac{1}{p(p+1)}\right) + O\left(x^{\frac{3}{2}} + \epsilon\right),$$

where ϵ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

On the other hand, similar to the famous Euler totient function $\varphi(n)$, Professor F.Russo defined a new arithmetical function called the Smarandache totient function $St(n) = \varphi(a(n))$, where $a(n)$ is any Smarandache sequence. Then he asked us to study the properties of these functions. At the same time, he proposed the following:

Conjecture. For the Smarandache power sequence $SP(k)$, $\frac{S_1}{S_2}$ converges to zero as $n \to \infty$, where $S_1 = \sum_{k=1}^{n} \left(\frac{1}{St(k)}\right)^2$, $S_2 = \left(\sum_{k=1}^{n} \frac{1}{St(k)}\right)^2$.

In this paper, we shall use the elementary and analytic methods to study this problem, and prove that the conjecture is correct. That is, we shall prove the following:

Theorem. For the Smarandache power function $SP(k)$, we have $\lim_{n\to\infty} \frac{S_1}{S_2} = 0$, where $S_1 = \sum_{k=1}^{n} \left(\frac{1}{\varphi(SP(k))}\right)^2$, $S_2 = \left(\sum_{k=1}^{n} \frac{1}{\varphi(SP(k))}\right)^2$.

§2. Some lemmas

To complete the proof of the theorem, we need the following two simple Lemmas:

Lemma 1. For any given real number $\epsilon > 0$, there exists a positive integer $N(\epsilon)$, such that for all $n \geq N(\epsilon)$, we have $\varphi(n) \geq (1-\epsilon)\frac{c \cdot n}{\ln \ln n}$, where c is a constant.

Proof. See reference [5].

Lemma 2. For the Euler totient fuction $\varphi(n)$, we have the asymptotic formula

$$\sum_{k \leq n} \frac{1}{\varphi(k)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \ln n + A + O\left(\frac{\ln n}{n}\right),$$

where $A = \gamma \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n\varphi(n)} - \sum_{n=1}^{\infty} \frac{\mu^2(n) \ln n}{n\varphi(n)}$ is a constant.

Proof. See reference [6].

§3. Proof of the theorem

In this section, we shall prove our Theorem.

We separate all integer k in the interval $[1, n]$ into two subsets A and B as follows: A : the set of all square-free integers. B : the set of other positive integers k such that $k \in [1, n]\backslash A$. So we have

$$\sum_{k \leq n} \frac{1}{(\varphi(SP(k)))^2} = \sum_{k \in A} \frac{1}{(\varphi(SP(k)))^2} + \sum_{k \in B} \frac{1}{(\varphi(SP(k)))^2}.$$

From the definition of the subset A, we may get

$$\sum_{k \in A} \frac{1}{(\varphi(SP(k)))^2} = \sum_{k \in A} \frac{1}{k^2 \prod_{p|k} \left(1 - \frac{1}{p}\right)^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2 \prod_{p|k} \left(1 - \frac{1}{p}\right)^2} \ll 1.$$

By Lemma 1, we can easily get $\frac{k}{\varphi(k)} = O(\ln\ln k)$. Note that $\sum\limits_{k\le n}\frac{\mu^2(k)}{k^2} = O(1)$. And if $k\in B$, then we can write k as $k = l\cdot m$, where l is a square-free integer and m is a square-full integer. Let S denote $\sum\limits_{k\in B}\frac{1}{(\varphi(SP(k)))^2}$, then from the properties of $SP(k)$ and $\varphi(k)$ we have

$$S \le \sum_{lm\le n}\frac{1}{l^2\prod\limits_{p|m}p^2\prod\limits_{p|lm}\left(1-\frac{1}{p}\right)^2} = \sum_{m\le n}\frac{1}{\prod\limits_{p|m}p^2}\sum_{l\le\frac{n}{m}}\frac{\mu^2(l)}{l^2}\cdot\frac{l^2m^2}{\varphi^2(lm)} = O\left((\ln\ln n)^2\sum_{m\le n}\frac{1}{\prod\limits_{p|m}p^2}\right).$$

Let $U(k) = \prod\limits_{p|k}p$, then $\sum\limits_{m\le n}\frac{1}{\prod\limits_{p|m}p^2} = \sum\limits_{k\le n}\frac{a(k)}{U^2(k)}$, where m is a square-full integer and the arithmetical function $a(k)$ is defined as follows:

$$a(k) = \begin{cases} 1, & \text{if } k \text{ is a square-full integer;}\\ 0, & \text{otherwise.}\end{cases}$$

Note that $\frac{a(k)}{U^2(k)}$ is a multiplicative function. According to the Euler product formula (see reference [3] and [5]), we have

$$A(s) \;=\; \sum_{k=1}^{\infty}\frac{a(k)}{U^2(k)k^s} = \prod_p\left(1+\frac{1}{p^{2+s}(p^s-1)}\right).$$

From the Perron formulas [5], for $b = 1+\frac{1}{\ln n}$, $T\ge 1$, we have

$$\sum_{k\le n}\frac{a(k)}{U^2(k)} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT}A(s)\frac{n^s}{s}ds + O\left(\frac{n^b\zeta(b)}{T}\right) + O\left(n\min\left(1,\frac{\ln n}{T}\right)\right) + \frac{a(n)}{2U^2(n)}.$$

Taking $T = n$, we can get the estimate

$$O\left(\frac{n^b\zeta(b)}{T}\right) + O\left(n\min\left(1,\frac{\ln n}{T}\right)\right) + \frac{a(n)}{2U^2(n)} = O(\ln n).$$

Because the function $A(s)\frac{n^s}{s}$ is analytic in $Re\ s > 0$, taking $c = \frac{1}{\ln n}$, then we have

$$\frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT}A(s)\frac{n^s}{s}ds + \int_{c-iT}^{b-iT}A(s)\frac{n^s}{s}ds + \int_{b+iT}^{c+iT}A(s)\frac{n^s}{s}ds + \int_{c+iT}^{c-iT}A(s)\frac{n^s}{s}ds\right) = 0.$$

Note that $\int_{c-iT}^{c+iT}A(s)\frac{n^s}{s}ds = O\left(\int_{-T}^{T}\frac{dy}{\sqrt{c^2+y^2}}\right) = O(\ln n)$ and $\int_{c-iT}^{b-iT}A(s)\frac{n^s}{s}ds = O\left(\int_c^b\frac{n^\sigma}{T}d\sigma\right) = O\left(\frac{1}{\ln n}\right)$. Similarly, $\int_{c+iT}^{b+iT}A(s)\frac{n^s}{s}ds = O\left(\frac{1}{\ln n}\right)$. Hence, $\sum\limits_{k\le n}\frac{a(k)}{U^2(k)} = O(\ln n)$.

So

$$\sum_{k\leq n}\frac{1}{(\varphi(SP(k)))^2}=O(\ln n\cdot(\ln\ln n)^2). \tag{1}$$

Now we come to estimate $\sum_{k\leq n}\frac{1}{\varphi(SP(k))}$, from the definition of $SP(n)$, we may immediately get that $SP(n)\leq n$. Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, then $SP(n)=p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s}$, where $\beta_i\geq 1$. Therefore, we can get that $p_1^{\beta_1-1}(p_1-1)p_2^{\beta_2-1}(p_2-1)\cdots p_s^{\beta_s-1}(p_s-1)\leq p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_s^{\alpha_s-1}(p_s-1)$, thus $\varphi(p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s})\leq\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s})$. That is, $\varphi(SP(n))\leq\varphi(n)$, according to Lemma 2, we can easily get

$$\sum_{k\leq n}\frac{1}{\varphi(SP(k))}\geq\sum_{k\leq n}\frac{1}{\varphi(k)}=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\ln n+A+O\left(\frac{\ln n}{n}\right). \tag{2}$$

Combining (1) and (2), we obtain

$$0\leq\frac{\sum_{k=1}^{n}\left(\frac{1}{\varphi(SP(k))}\right)^2}{\left(\sum_{k=1}^{n}\frac{1}{\varphi(SP(k))}\right)^2}\leq\frac{O\left(\ln n\cdot(\ln\ln n)^2\right)}{\left(\frac{\zeta(2)\zeta(3)}{\zeta(6)}\ln n+A+O\left(\frac{\ln n}{n}\right)\right)^2}\longrightarrow 0,\quad \text{as } n\to\infty.$$

This completes the proof of our Theorem.

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