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Smarandachely t-path step signed graphs

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Abstract A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph.* E. Prisner [9] in his book Graph Dynamics defines the t-path step operator on the class of finite graphs. Given a graph G and a positive integer t, the *t-path step graph* $(G)_t$ of G is formed by taking a copy of the vertex set $V(G)$ of G, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there exists a $u - v$ path of length t in G. Analogously, one can define the *Smarandachely t-path step signed graph* $(S)_t = ((G)_t, \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $(G)_t$ called *t-path step graph* and sign of any edge $e = uv$ in $(S)_t$ is $\mu(u)\mu(v)$. It is shown that for any signed graph S, its $(S)_t$ is balanced. We then give structural characterization of Smarandachely t-path step signed graphs. Further, in this paper we characterize signed graphs which are switching equivalent to their Smarandachely 3-path step signed graphs.

Keywords Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, Smarandachely t-path step signed graphs, negation.

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [4]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph.* A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (Harary [3]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ by S_μ. Given a signed graph S one can easily define a marking μ of S as follows:

For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{u \in N(v)} \sigma(uv),$$

the marking μ of S is called *canonical* marking of S.

The following characterization of balanced signed graphs is well known.

Proposition 1.1.[6] A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.

Given a marking μ of S, by *switching* S with respect to μ we mean reversing the sign of every edge of S whenever the end vertices have opposite signs in $\mathcal{S}_\mu$[1] . We denote the signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and this signed graph is called the *μ-switched signed graph* or just *switched signed graph*. A signed graph S_1 switches to a signed graph S_2 (that is, they are *switching equivalent* to each other), written $S_1 \sim S_2$, whenever there exists a marking μ such that $\mathcal{S}_\mu(S_1) \cong S_2$.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (Sozánsky [7]) or *cycle isomorphic* (Zaslavsky [8]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known:

Proposition 1.2.[8] Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

§2. Smarandachely t-path step signed graphs

Given a graph G and a positive integer t, the *t-path step graph* $(G)_t$ of G is formed by taking a copy of the vertex set $V(G)$ of G, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there exists a $u - v$ path of length t in G. The notion of *t-path step graphs* was defined in [9], page 168.

In this paper, we shall now introduce the concept of Smarandachely t-path step signed graphs as follows: The *Smarandachely t-path step signed graph* $(S)_t = ((G)_t, \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $(G)_t$ called *t-path step graph* and sign of any edge $e = uv$ in $(S)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called *Smarandachely t-path step signed graph*, if $S \cong (S')_t$, for some signed graph S'.

The following result indicates the limitations of the notion of Smarandachely t-path step signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be Smarandachely t-path step signed graphs.

Proposition 2.1. For any signed graph $S = (G, \sigma)$, its $(S)_t$ is balanced.

Proof. Since sign of any edge $e = uv$ is $(S)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1.1, $(S)_t$ is balanced.

Remark. For any two signed graphs S and S' with same underlying graph, their Smarandachely t-path step signed graphs are switching equivalent.

Corollary 2.2. For any signed graph $S = (G, \sigma)$, its Smarandachely 2 (3)-path step signed graph $(S)_2$ $((S)_3)$ is balanced.

The following result characterize signed graphs which are Smarandachely t-path step signed graphs.

Proposition 2.3. A signed graph $S = (G, \sigma)$ is a Smarandachely t-path step signed graph if, and only if, S is balanced signed graph and its underlying graph G is a t-path step graph.

Proof. Suppose that S is balanced and G is a t-path step graph. Then there exists a graph H such that $(H)_t \cong G$. Since S is balanced, by Proposition 1.1, there exists a marking μ of G such that each edge $e = uv$ in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $(S')_t \cong S$. Hence S is a Smarandachely t-path step signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a Smarandachely t-path step signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $(S')_t \cong S$. Hence G is the t-path step graph of H and by Proposition 2.1, S is balanced.

§3. Switching invariant Smarandachely 3-path step signed graphs

Zelinka [9] prove hat the graphs in Fig. 1 are all unicyclic graphs which are fixed in the operator $(G)_3$, i.e. graphs G such that $G \cong (G)_3$. The symbols p, q signify that the number of vertices and edges in Fig. 1.

Proposition 3.1.[9] Let G be a finite unicyclic graph such that $G \cong (G)_3$. Then either G is a circuit of length not divisible by 3, or it is some of the graphs depicted in Fig. 1.

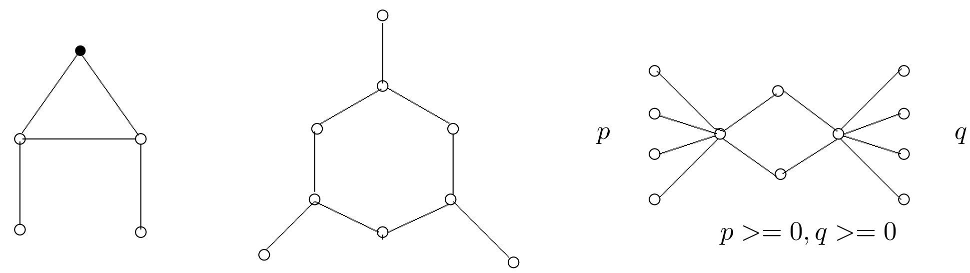

Fig.1.

In view of the above result, we have the following result for signed graphs:

Proposition 3.2. For any signed graph $S = (G, \sigma)$, $S \sim (S)_3$ if, and only if, G is a cycle of length not divisible by 3, or it is some of the graphs depicted in Fig. 1 and S is balanced.

Proof. Suppose $S \sim (S)_3$. This implies, $G \cong (G)_3$ and hence by Proposition 3.1, we see that the G must be isomorphic to either C_m, $4 \leq m \neq 3k$, k is a positive integer or the graphs depicted in Fig. 1. Now, if S is any signed graph on any of these graphs, Corollary 4 implies that $(S)_3$ is balanced and hence if S is unbalanced its Smarandachely 3-path step signed graph $(S)_3$ being balanced cannot be switching equivalent to S in accordance with Proposition 1.2. Therefore, S must be balanced.

Conversely, suppose that S is a balanced signed graph on C_m, $4 \leq m \neq 3k$, k is a positive integer or the graphs depicted in Fig. 1. Then, since $(S)_3$ is balanced as per Corollary 2.2 and since $G \cong (G)_3$ in each of these cases, the result follows from Proposition 1.2.

Problem. Characterize the signed graphs for which $S \cong (S)_3$.

The notion of *negation* $\eta(S)$ of a given signed graph S defined by Harary [3] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $(S)_t$ is balanced (Proposition 2.1). We now examine, the condition under which negation of $(S)_t$ (i.e., $\eta((S)_t)$) is balanced.

Proposition 3.3. Let $S = (G, \sigma)$ be a signed graph. If $(G)_t$ is bipartite then $\eta((S)_t)$ is balanced.

Proof. Since, by Proposition 2.1, $(S)_t$ is balanced, then every cycle in $(S)_t$ contains even number of negative edges. Also, since $(G)_t$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $(S)_t$ are also even. This implies that the same thing is true in negation of $(S)_t$. Hence $\eta((S)_t)$ is balanced.

Proposition 3.2 provides easy solutions to three other signed graph switching equivalence relations, which are given in the following results.

Corollary 3.4. For any signed graph $S = (G, \sigma)$, $\eta(S) \sim (S)_3$ if, and only if, S is unbalanced signed graph on C_{2m+1}, $m \geq 2$ or first two graphs depicted in Fig. 1.

Corollary 3.5. For any signed graph $S = (G, \sigma)$, $(\eta(S))_3 \sim (S)_3$.

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