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Smarandache U-liberal semigroup structure

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Abstract In this paper, Smarandache U-liberal semigroup structure is given. It is shown that a semigroup S is Smarandache U-liberal semigroup if and only if it is a strong semilattice of some rectangular monoids. Consequently, some corresponding results on normal orthocryptou semigroups and normal orthocryptogroups are generalized and extended.

Keywords Smarandache U-liberal semigroup, $U-$semiabundant semigroups, normal band, rectangular monoid, strong semilattice.

§1. Introduction and preliminaries

In order to generalize regular semigroups, new Green's relations, namely, the Green's $*$-relations on a semigroup S have been introduced in [1] and [2] as follows:

$$\mathcal{L}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)ax = ay \Leftrightarrow bx = by\},$$

$$\mathcal{R}^* = \{(a, b) \in S \times S : (\forall x, y \in S^1)xa = ya \Leftrightarrow xb = yb\},$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*, \qquad \mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*.$$

In [3], Fountain investigated a class of semigroups called abundant semigroups in which each $\mathcal{L}^*-$class and each $\mathcal{R}^*-$class of S contain at least an idempotent. Actually, the class of regular semigroups are properly contained in the class of abundant semigroups.

In 1980, El-Qallali generalized the Green's $*$- relations to the Green's $\sim -$relations on a semigroup S in [4] as follows:

$$\tilde{\mathcal{L}} = \{(a, b) \in S \times S : (\forall e \in E(S))\ ae = a \Leftrightarrow be = b\},$$

$$\tilde{\mathcal{R}} = \{(a, b) \in S \times S : (\forall e \in E(S))\ ea = a \Leftrightarrow eb = b\},$$

$$\tilde{\mathcal{H}} = \tilde{\mathcal{L}} \cap \tilde{\mathcal{R}}, \qquad \tilde{\mathcal{D}} = \tilde{\mathcal{L}} \vee \tilde{\mathcal{R}}.$$

In his thesis, El-Qallali obtained and studied a much bigger class of semigroups, called semi-abundant semigroup.

After that, many authors study this class of semigroups, and obtain a lot of interesting conclusions and results(see [5],[6],[7] etc.).

In recent years, some scholars have observed that one can pay special attention to a subset U of $E(S)$ instead of the whole set $E(S)$ of a semiabundant semigroup S. In particular, Lawson in [8] noticed that if U is a subset of $E(S)$ of a semiabundant semigroup S then U is perhaps good enough to provide sufficient information for the whole semigroup S. The semigroup S is usually denoted by $S(U)$ and the equivalence relations on $S(U)$ with respect to $U \subseteq E(S)$ can be given by

$$\tilde{\mathcal{L}}^U = \{(a,b) \in S \times S | U_a^r = U_b^r\},$$
$$\tilde{\mathcal{R}}^U = \{(a,b) \in S \times S | U_a^l = U_b^l\},$$
$$\tilde{\mathcal{H}}^U = \tilde{\mathcal{L}}^U \cap \tilde{\mathcal{R}}^U,$$
$$\tilde{\mathcal{Q}}^U = \{(a,b) \in S \times S | U_a = U_b\},$$

where $U_a^l = \{u \in U | ua = a\}$, $U_a^r = \{u \in U | au = a\}$ and $U_a = U_a^l \cap U_a^r = \{u \in U | ua = a = au\}$ for any $a \in S$.

A semigroup $S(U)$ is said to be a $U-$ semiabundant semigroup if every $\tilde{\mathcal{L}}^U$ and every $\tilde{\mathcal{R}}^U-$ class of $S(U)$ contain at least one element of U respectively. A semigroup $S(U)$ is said to be a $U-$ semi-superabundant semigroup if every $\tilde{\mathcal{H}}^U$ of $S(U)$ contains at least one element of U. In this case, the unique element in $\tilde{\mathcal{H}}_a^U \cap U$ is denoted by $a_U^{\diamond}$. On the other hand, a semigroup $S(U)$ is called by He in [7] a $U-$ liberal semigroup if every $\tilde{\mathcal{Q}}^U-$ class of S contains an element of U. It is routine to check that a $\tilde{\mathcal{Q}}^U-$ class contains at most one element of U. Denote the unique element in $\tilde{\mathcal{Q}}_a^U \cap U$, if it exists, by a_U° . The structure of Smarandache $U-$ liberal semigroups has also been recently investigated by He in [7].

For a Smarandache $U-$ liberal semigroup $S(U)$, we call the following condition the Ehresmann type condition, in brevity, the ET-condition:

$$(\forall a, b \in S)(ab)_U^0 \mathcal{D}(U) a_U^0 b_U^0,$$

where

$$\mathcal{D}(U) = \{(e,f) \in U \times U | (\exists g \in U) e\mathcal{R}g\mathcal{L}f\}.$$

A Smarandache $U-$ liberal semigroup $S(U)$ is called an orthodox $U-$ liberal semigroup if U is a subsemigroup of $S(U)$ and the ET-condition holds on $S(U)$.

In general, unlike the usual Green's relations on a semigroup S, $\tilde{\mathcal{L}}^U$ is not necessarily a right congruence on S and $\tilde{\mathcal{R}}^U$ is not necessarily a left congruence on S (see [8]).

We say that a semigroup $S(U)$ satisfies the (CR) condition if $\tilde{\mathcal{L}}^U$ is a right congruence on S and that $S(U)$ satisfies the (CL) condition if $\tilde{\mathcal{R}}^U$ is a left congruence on S. If the semigroup $S(U)$ satisfies both the (CR) and (CL) condition, then we say $S(U)$ satisfies the (C) condition.

The studies on the structures of semigroups play an important role in the research of the algebraic theories of semigroups. From [7], it is known that a $U-$semi-superabundant semigroup $S(U)$ is an orthodox $U-$liberal semigroup for some $U \subseteq E(S)$ if and only if it is a semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(U_\alpha)]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha \in Y} U_\alpha$ is a subsemigroup of S. Meanwhile, notice that a normal band is a strong semilattice of some rectangular bands. Naturally, we will quote such a question: whether will a normal orthodox U-liberal semigroup $S(U)$ be a strong semilattice of some rectangular monoids?

In this paper, we will consider the question quoted above. Consequently, we show that a semigroup $S(U)$ is a normal orthodox Smarandache $U-$liberal semigroup if and only if it is a strong semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(U_\alpha); \Phi_{\alpha,\beta}]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha\in Y} U_\alpha$ is a normal band of $S(U)$. Consequently, some corresponding results on normal orthocryptou semigroups and normal orthocryptogroups are generalized and extended.

For notations and terminologies not mentioned in this paper, the reader is referred to [7],[9],[10].

§2. Normal Orthodox $U-$liberal Semigroups

In this section, we will give a construction of normal orthodox $U-$liberal semigroups.

Firstly, we recall the following lemmas.

Lemma 2.1. [7] Let $\mathcal{F}$ be one of Green's relations $\mathcal{L}, \mathcal{R}$ or $\mathcal{H}$ and $\tilde{\mathcal{F}}^U$ the corresponding Green $\sim -$ relations on the semigroup S. Then, for any $a, b \in S$, we have

(i) $\mathcal{F} \subseteq \tilde{\mathcal{F}}^U$and for $a, b \in Reg_U(S)$, $a, b \in \tilde{\mathcal{F}}^U$ if and only if $a, b \in \mathcal{F}$, where $Reg_U(S) = \{a \in S|(\exists e, f \in U)e\mathcal{L}a\mathcal{R}f\}$;

(ii)$\tilde{\mathcal{H}}^U \subseteq \tilde{\mathcal{Q}}^U$ and $\tilde{Q}^U_a$ contains at most one element in U;

(iii) If $S(U)$ is a $U-$ semi-superabundant semigroup, then $S(U)$ is a Smarandache $U-$ liberal semigroup with $\tilde{\mathcal{Q}}^U = \tilde{\mathcal{H}}^U$.

Lemma 2.2. [7] The following statements are equivalent for a semigroup S:

(i) $S(U)$ is a Smarandache $U-$ liberal semigroup for some $U \subseteq E(S)$ and U itself is a rectangular band;

(ii) $S(U)$ is an orthodox $U-$ liberal semigroup such that U is a rectangular band;

(iii) S is isomorphic to a rectangular monoid.

Lemma 2.3. [7] The following statements are equivalent for a semigroup S:

(i) $S(U)$ is an orthodox U-liberal semigroup for some $U \subseteq E(S)$;

(ii) $S = [Y; S_\alpha(U_\alpha)]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha\in Y} U_\alpha$ is a subsemigroup of S;

(iii)$S(U)$ is a U-semi-superabundant semigroup satisfying the (C) condition for some $U \subseteq E(S)$ and U is a subsemigroup of S.

Now, we will give our main theorem.

Theorem 2.4. The following statements are equivalent for a semigroup S:

(i) $S(U)$ is a normal orthodox U-liberal semigroup for some normal band $U \subseteq E(S)$;

(ii) $S(U)$ is a strong semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(U_\alpha); \Phi_{\alpha,\beta}]$, where $S_\alpha(U_\alpha)$ is a rectangular monoid for every $\alpha \in Y$ and $U = \cup_{\alpha\in Y} U_\alpha$ is a normal band of S;

(iii)$S(U)$ is a U-semi-superabundant semigroup satisfying the (C) condition for some $U \subseteq E(S)$ and U is a normal band of S.

Proof. $(i) \Rightarrow (ii)$

Assume that $S(U)$ is a normal orthodaox $U-$liberal semigroup for some normal band $U \subseteq E(S)$. Note that $U \subseteq E(S)$ is a normal band, we will have $U = [Y; U_\alpha, f_{\alpha,\beta}]$ or $U = [Y; I_\alpha \times \Lambda_\alpha; \varphi_{\alpha,\beta}, \psi_{\alpha,\beta}]$, where $(i,j)f_{\alpha,\beta} = (i\varphi_{\alpha,\beta}, j\psi_{\alpha,\beta}), (i,j) \in I_\alpha \times \Lambda_\alpha$.

For any $\alpha \in Y$, we form the set $S_\alpha = \{x \in S | x_U^0 \in U_\alpha\}$. Since $S(U)$ satisfies the ET-condition, for all $x \in S_\alpha$, $y \in S_\beta$, we have $(xy)_U^0 \mathcal{D}(U) x_U^0 y_U^0$. This leads to $xy \in S_{\alpha\beta}$ and hence $S(U) = [Y; S_\alpha(U_\alpha)]$.

Notice that every semigroup $S_\alpha(U_\alpha)$ is a Smarandache $U_\alpha-$ liberal semigroup and U_α is a rectangular band. By Lemma 2.2, $S_\alpha(U_\alpha)$ is isomorphic to a rectangular monoid. For convenience, we denote $S_\alpha(U_\alpha) = I_\alpha \times T_{U_\alpha} \times \Lambda_\alpha$.

Now, define a mapping,

$$\Phi_{\alpha,\beta} : S_\alpha(U_\alpha) \to S_\beta(U_\beta),$$

$$(i_\alpha, u_\alpha, \lambda_\alpha) \to (i_\alpha\varphi_{\alpha,\beta}, 1_{U_\beta}, \lambda_\alpha\psi_{\alpha,\beta})(i_\alpha, u_\alpha, \lambda_\alpha) = (i_\alpha\varphi_{\alpha,\beta}i_\alpha, 1_{U_\beta}u_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha).$$

In the following, we will prove that S is a strong semilattice of $S_\alpha(U_\alpha)$, i.e., $S = [Y; S_\alpha(U_\alpha); \Phi_{\alpha,\beta}]$.

Firstly, $\Phi_{\alpha,\beta}$ is a homomorphism.

For any $x = (i_\alpha, u_\alpha, \lambda_\alpha)$, $y = (j_\alpha, v_\alpha, \mu_\alpha) \in S_\alpha (\forall \alpha \geq \beta)$,

$$\begin{aligned} (xy)\Phi_{\alpha,\beta} &= [(i_\alpha, u_\alpha, \lambda_\alpha)(j_\alpha, v_\alpha, \mu_\alpha)]\Phi_{\alpha,\beta} \\ &= (i_\alpha j_\alpha, u_\alpha v_\alpha, \lambda_\alpha\mu_\alpha)\Phi_{\alpha,\beta} \\ &= (i_\alpha, u_\alpha v_\alpha, \mu_\alpha)\Phi_{\alpha,\beta} \\ &= (i_\alpha\varphi_{\alpha,\beta}i_\alpha, 1_{U_\beta}u_\alpha v_\alpha, \mu_\alpha\psi_{\alpha,\beta}\mu_\alpha), \end{aligned}$$

$$\begin{aligned} x\Phi_{\alpha,\beta}y\Phi_{\alpha,\beta} &= (i_\alpha, u_\alpha, \lambda_\alpha)\Phi_{\alpha,\beta}(j_\alpha, v_\alpha, \mu_\alpha)\Phi_{\alpha,\beta} \\ &= (i_\alpha\varphi_{\alpha,\beta}i_\alpha, 1_{U_\beta}u_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)(j_\alpha\varphi_{\alpha,\beta}j_\alpha, 1_{U_\beta}v_\alpha, \mu_\alpha\psi_{\alpha,\beta}\mu_\alpha) \\ &= (i_\alpha\varphi_{\alpha,\beta}i_\alpha j_\alpha\varphi_{\alpha,\beta}j_\alpha, 1_{U_\beta}u_\alpha 1_{U_\beta}v_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\mu_\alpha\psi_{\alpha,\beta}\mu_\alpha) \\ &= (i_\alpha\varphi_{\alpha,\beta}i_\alpha, 1_{U_\beta}u_\alpha v_\alpha, \mu_\alpha\psi_{\alpha,\beta}\mu_\alpha) \quad (\text{ since } U_\alpha \text{ is normal}). \end{aligned}$$

Thus, $(xy)\Phi_{\alpha,\beta} = x\Phi_{\alpha,\beta}y\Phi_{\alpha,\beta}$.

Secondly, $\Phi_{\alpha,\alpha}$ is an identity mapping.

For any $x = (i_\alpha, u_\alpha, \lambda_\alpha) \in S_\alpha$, $x\Phi_{\alpha,\alpha} = (i_\alpha\varphi_{\alpha,\alpha}i_\alpha, 1_{U_\alpha}u_\alpha, \lambda_\alpha\psi_{\alpha,\alpha}\lambda_\alpha) = (i_\alpha, u_\alpha, \lambda_\alpha) = x$. Hence, $\Phi_{\alpha,\alpha}$ is an identity mapping.

Thirdly, notice that for any $x = (i_\alpha, 1_{U_\alpha}, \lambda_\alpha) \in E(S_\alpha) = I_\alpha \times 1_{U_\alpha} \times \Lambda_\alpha$, $y = (i_\beta, 1_{U_\beta}, \lambda_\beta) \in E(S_\beta) = I_\beta \times 1_{U_\beta} \times \Lambda_\beta$, $xy \in E(S_{\alpha\beta}) = I_{\alpha\beta} \times 1_{U_{\alpha\beta}} \times \Lambda_{\alpha\beta}$, we can get $1_{U_\alpha}1_{U_\beta} = 1_{U_{\alpha\beta}}$. Especially, when $\alpha \geq \beta$, we have $1_{U_\alpha}1_{U_\beta} = 1_{U_{\alpha\beta}} = 1_{U_\beta}$. Now, for any $\alpha, \beta, \gamma \in Y (\alpha \geq \beta \geq \gamma)$ and any $x = (i_\alpha, u_\alpha, \lambda_\alpha) \in S_\alpha(U_\alpha)$, we will have

$$(x\Phi_{\alpha,\beta})\Phi_{\beta,\gamma} = (i_\alpha, u_\alpha, \lambda_\alpha)\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} = (i_\alpha\varphi_{\alpha,\beta}i_\alpha, 1_{U_\beta}u_\alpha, \lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\Phi_{\beta,\gamma}$$

$$\begin{aligned} &= ((i_\alpha\varphi_{\alpha,\beta}i_\alpha)\varphi_{\alpha,\beta}(i_\alpha\varphi_{\alpha,\beta}i_\alpha), 1_{U_\gamma}(1_{U_\beta}u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\ &= (((i_\alpha\varphi_{\alpha,\beta})(i_\alpha\varphi_{\alpha,\beta}i_\alpha))\varphi_{\alpha,\beta}(i_\alpha\varphi_{\alpha,\beta}i_\alpha), 1_{U_\gamma}(1_{U_\beta}u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\ &= (i_\alpha\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}(i_\alpha\varphi_{\alpha,\beta}i_\alpha)\varphi_{\beta,\gamma}(i_\alpha\varphi_{\alpha,\beta}i_\alpha), 1_{U_\gamma}(1_{U_\beta}u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\ &= (i_\alpha\varphi_{\alpha,\gamma}(i_\alpha\varphi_{\alpha,\beta}i_\alpha)\varphi_{\beta,\gamma}(i_\alpha\varphi_{\alpha,\beta})i_\alpha, 1_{U_\gamma}(1_{U_\beta}u_\alpha), (\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)\psi_{\beta,\gamma}(\lambda_\alpha\psi_{\alpha,\beta}\lambda_\alpha)) \\ &= (i_\alpha\varphi_{\alpha,\gamma}i_\alpha, 1_{U_\gamma}u_\alpha, \lambda_\alpha\psi_{\alpha,\gamma}\lambda_\alpha) \quad (\text{since } I_\alpha \text{ is a left zero band}) \\ &= x\Phi_{\alpha,\gamma}. \end{aligned}$$

Finally, for any $\alpha,\beta\in Y$, $a_\alpha\in S_\alpha$ and $b_\beta\in S_\beta$, since $a_\alpha b_\beta\in S_{\alpha\beta}\triangleq S_\gamma$, we have

$$\begin{aligned} a_\alpha b_\beta &= (i_\alpha,u_\alpha,\lambda_\alpha)(i_\beta,u_\beta,\lambda_\beta)\\ &= (i_\alpha i_\beta,u_\alpha u_\beta,\lambda_\alpha\lambda_\beta)\\ &= (i_\alpha\varphi_{\alpha,\gamma}i_\alpha i_\beta\varphi_{\beta,\gamma}i_\beta,1_{U_\gamma}u_\alpha 1_{U_\gamma}u_\beta,\lambda_\alpha\psi_{\alpha,\gamma}\lambda_\alpha\lambda_\beta\psi_{\beta,\gamma}\lambda_\beta)\\ &= (i_\alpha\varphi_{\alpha,\gamma}i_\alpha,1_{U_\gamma}u_\alpha,\lambda_\alpha\psi_{\alpha,\gamma}\lambda_\alpha)(i_\beta\varphi_{\beta,\gamma}i_\beta,1_{U_\gamma}u_\beta,\lambda_\beta\psi_{\beta,\gamma}\lambda_\beta)\\ &= (a_\alpha\Phi_{\alpha,\gamma})(b_\beta\Phi_{\beta,\gamma}). \end{aligned}$$

Thus, summing up the above discussions, $S(U)$ is isomorphic to a strong semilattice of rectangular monoids $S_\alpha(U_\alpha)$, that is, $S(U)=[Y;S_\alpha(U_\alpha);\Phi_{\alpha,\beta}]$.

$(ii)\Rightarrow(iii)$ The proof is similar with the corresponding $(ii)\Rightarrow(iii)$ of Lemma 2.3.

For any $\alpha\in Y$, assume that $S_\alpha(U_\alpha)=I_\alpha\times T_\alpha\times\Lambda_\alpha$. Then, it is not hard to see that for any $(i,x,\lambda)\in S_\alpha$,$(j,y,\mu)\in S_\beta$, $(i,x,\lambda)\tilde{\mathcal{L}}^U(j,y,\mu)$ if and only if $\alpha=\beta$ and $\lambda=\mu\in\Lambda_\alpha$. On the other hand, if $(i,x,\lambda)\tilde{\mathcal{L}}^U(j,y,\lambda)$ for some $\lambda\in\Lambda_\alpha$, then for all $(k,z,\nu)\in S_\gamma(\nu\in Y)$, we have

$$(i,1_{T_\alpha},\lambda)(k,z,\nu)=(k',z',\nu')(\in S_{\alpha\gamma}),$$

$$(i,x,\lambda)(k',1_{T_{\alpha\gamma}},\nu')=(i',x',\lambda'),$$

$$(j,y,\lambda)(k',1_{T_{\alpha\gamma}},\nu')=(j',y',\lambda''),$$

Consequently, by using the above relations, we derive that
$(i,x,\lambda)(k,z,\nu)=(i,x,\lambda)(i,1_{T_\alpha},\lambda)(k,z,\nu)=(i,x,\lambda)(k',z',\nu')=(i,x,\lambda)(k',1_{T_{\alpha\gamma}},\nu')(k',z',\nu)'$ $=(i',x',\lambda')(k',z',\nu')$= $(i',x'z',\lambda')$;
$(j,y,\lambda)(k,z,\nu)=(j,y,\lambda)(i,1_{T_\alpha},\lambda)(k,z,\nu)=(j,y,\lambda)(k',z',\nu')=(j,y,\lambda)(k',1_{T_{\alpha\gamma}},\nu')(k',z',\nu)'$ $=(j',y',\lambda'')(k',z',\nu')$= $(j',y'z',\nu')$.
Thereby, we obtain that $(i,x,\lambda)(k,z,\nu)\tilde{\mathcal{L}}^U(j,y,\lambda)(k,z,\nu)$ so that $\tilde{\mathcal{L}}^U$ is a right congruence on S.

Similarly, we can show that $(i,x,\lambda)\tilde{\mathcal{R}}^U(j,y,\lambda)$ if and only if $i=j\in I_\alpha$ for some $\alpha\in Y$,and so $\tilde{\mathcal{R}}^U$ is a left congruence on S.

Hence, together with Lemma 2.3, $S(U)$ is a U-semi-superabundant semigroup satisfying the (C) condition for some $U\subseteq E(S)$. Note that U is a normal band of S, (iii) holds.

$(iii)\Rightarrow(i)$ The proof is similar with the corresponding $(iii)\Rightarrow(i)$ of Lemma 2.3.

Assume that (iii)holds. Then, by Lemma 2.1, $S(U)$ is Smarandache $U-$ liberal semigroup and for all $a\in S(U)$, $a^\circ_U=a^\diamond_U$. Since $S(U)$ satisfies the (C) condition, we have, for all $a,b\in S(U)$,

$$(ab)^\diamond_U\tilde{\mathcal{R}}^U ab\tilde{\mathcal{R}}^U ab^\diamond_U\tilde{\mathcal{R}}^U(ab^\diamond_U)^\diamond_U\tilde{\mathcal{L}}^U ab^\diamond_U\tilde{\mathcal{L}}^U a^\diamond_U b^\diamond_U.$$

This leads to $(ab)^\diamond_U\tilde{\mathcal{R}}^U(ab^\diamond_U)^\diamond_U\tilde{\mathcal{L}}^U a^\diamond_U b^\diamond_U$. By Lemma 2.1 (i), we will get $(ab)^\diamond_U\mathcal{R}(ab^\diamond_U)^\diamond_U\mathcal{L}a^\diamond_U b^\diamond_U$. Consequently, $(ab)^\circ_U=(ab)^\diamond_U\mathcal{D}a^\diamond_U b^\diamond_U=a^\circ_U b^\circ_U$ holds. This shows that $S(U)$ satisfies the ET-condition. Note that U is a normal band of S, (i) holds.

Now, if we let $U=E(S)$ in Theorem 2.4, then we immediately have the following corollary.

Corollary 2.5. The following statements are equivalent for a semigroup S:

(i) $S(U)$ is a normal orthodox $E(S)$-liberal semigroup;

(ii) $S(U)$ is a strong semilattice of some rectangular monoids, i.e., $S = [Y; S_\alpha(E(S_\alpha)); \Phi_{\alpha,\beta}]$, where $S_\alpha(E(S_\alpha))$ is a rectangular monoid for every $\alpha \in Y$ and $E(S)$ is a normal band of S.

(iii)$S(U)$ is a semi-superabundant semigroup satisfying the (C) condition, and $E(S)$ is a normal band of S.

In the above corollary, if we restrict the semigroup S to the abundant or regular semigroups, then it is not hard for us to get

Corollary 2.6. The following statements are equivalent for a semigroup S:

(i) S is a normal orthocrypto semigroup;

(ii)S is a strong semilattice of rectangular cancellative monoids, i.e., $S = [Y; S_\alpha; \Phi_{\alpha,\beta}]$, where $S_\alpha = I_\alpha \times T_\alpha \times \Lambda_\alpha$, and I_α is a left zero band, Λ_α is a right zero band, T_α is a cancellative monoid for every $\alpha \in Y$.

Corollary 2.7. The following statements are equivalent for a semigroup S:

(i) S is a normal orthocryptogroup;

(ii)S is a strong semilattice of rectangular groups.

Hence, our main result generalizes and extends some corresponding results on normal orthocryptou semigroups and normal orthocryptogroups.

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