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On the value distribution properties of the Smarandache double-factorial function

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Abstract For any positive integer n, the famous Smarandache double-factorial function $SDF(n)$ is defined as the smallest positive integer m, such that $m!!$ is divisible by n, where the double factorial $m!! = 1 \cdot 3 \cdot 5 \cdots m$, if m is odd; and $m!! = 2 \cdot 4 \cdot 6 \cdots m$, if m is even. The main purpose of this paper is using the elementary and analytic methods to study the value distribution properties of $SDF(n)$, and give an interesting mean value formula for it.

Keywords The Smarandache double-factorial function, value distribution, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous Smarandache double-factorial function $SDF(n)$ is defined as the smallest positive integer m, such that $m!!$ is divisible by n, where the double factorial

$$m!! = \begin{cases} 1 \cdot 3 \cdot 5 \cdots m, & \text{if } m \text{ is odd ;} \\ 2 \cdot 4 \cdot 6 \cdots m, & \text{if } m \text{ is even.} \end{cases}$$

For example, the first few values of $SDf(n)$ are:

$$SDF(1) = 1,\ SDF(2) = 2,\ SDF(3) = 3,\ SDF(4) = 4,\ SDF(5) = 5,\ SDF(6) = 6,$$
$$SDF(7) = 7,\ SDF(8) = 4, SDF(9) = 9,\ SDF(10) = 10,\ SDF(11) = 11,\ SDF(12) = 6,$$
$$SDF(13) = 13,\ SDF(14) = 14,\ SDF(15) = 5,\ SDF(16) = 6\ \cdots\cdots.$$

In reference [1] and [2], F.Smarancdache asked us to study the properties of $SDF(n)$. About this problem, some authors had studied it, and obtained some interesting results, see reference [3]. In an unpublished paper, Zhu Minhui proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n \le x} SDF(n) = \frac{5\pi^2}{48} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{a_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where a_i are computable constants.

The other contents related to the Smarandache double-factorial function can also be found in references [4], [5], [6] and [7]. For example, Dr. Xu Zhefeng [4] studied the value distribution problem of the F.Smarandache function $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(S(n)-P(n))^2=\frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

The main purpose of this paper is using the elementary and analytic methods to study the value distribution problem of the double-factorial function $SDF(n)$, and give an interesting asymptotic formula it. That is, we shall prove the following conclusion:

Theorem 1. For any real number $x > 1$ and any fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x}(SDF(n)-P(n))^2=\frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $P(n)$ denotes the largest prime divisor of n, and all c_i are computable constants.

Now we define another function $S(n)$ as follows: Let $S(n)$ denotes the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : n|m!\}$. It is called the F.Smarandache function. For this function, using the method of proving Theorem 1 we can also get the following:

Theorem 2. For any real number $x > 1$ any fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x}(SDF(n)-S(n))^2=\frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1} x}\right).$$

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. We separate all integers n in the interval $[1,\ x]$ into two subsets A and B as follows: $A = \{n : 1 \leq n \leq x,\ P(n) > \sqrt{n}\}$; $B = \{n : 1 \leq n \leq x,\ n \notin A\}$, where $P(n)$ denotes the largest prime divisor of n. If $n \in A$, then $n = m \cdot P(n)$ and $P(m) < P(n)$. So from the definition of A we have $SDF(2) = 2$. For any positive integer $n > 2$ and $n \in A$, $SDF(n) = P(n)$, if $2 \dagger n$.

$SDF(n) = 2P(n)$, if $2 \mid n$. From this properties we have

$$\begin{aligned}
&\sum_{\substack{n\le x\\ n\in A}} (SDF(n)-P(n))^2\\
=&\sum_{\substack{2n\le x\\ 2n\in A}} (SDF(2n)-P(2n))^2+\sum_{\substack{2n-1\le x\\ 2n-1\in A}} (SDF(2n-1)-P(2n-1))^2\\
=&\sum_{\substack{n\le \frac{x}{2}\\ 2n\in A}} (SDF(2n)-P(2n))^2=\sum_{\substack{1<n\le \frac{x}{2}\\ 2n\in A}} (2P(2n)-P(2n))^2\\
=&\sum_{\substack{1<n\le \frac{x}{2}\\ 2n\in A}} P^2(2n)=\sum_{\substack{np\le \frac{x}{2}\\ p>2n}} p^2=\sum_{n\le \frac{\sqrt{x}}{2}}\sum_{2n<p\le \frac{x}{2n}} p^2. \qquad (1)
\end{aligned}$$

By the Abel's summation formula (See Theorem 4.2 of [8]) and the Prime Theorem (See Theorem 3.2 of [9]):

$$\pi(x)=\sum_{i=1}^{k}\frac{a_i\cdot x}{\ln^i x}+O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2,\cdots,k)$ are constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{2n<p\le \frac{x}{2n}} p^2 &= \frac{x^2}{(2n)^2}\cdot\pi\left(\frac{x}{2n}\right)-(2n)^2\cdot\pi(2n)-2\int_{2n}^{\frac{x}{2n}} y\cdot\pi(y)dy\\
&= \frac{x^3}{24n^3\ln x}+\sum_{i=2}^{k}\frac{b_i\cdot x^3\cdot\ln^i n}{n^3\cdot\ln^i x}+O\left(\frac{x^3}{n^3\cdot\ln^{k+1} x}\right), \qquad (2)
\end{aligned}$$

where we have used the estimate $2n \le \sqrt{x}$, and all b_i are computable constants.

Note that $\sum_{n=1}^{\infty}\frac{1}{n^3}=\zeta(3)$, from (1) and (2) we have

$$\sum_{\substack{n\le x\\ n\in A}} (SDF(n)-P(n))^2=\frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1} x}\right), \qquad (3)$$

where all c_i are computable constants.

For any positive integer n with $n \in B$, it is clear that $SDF(n) \ll \sqrt{n}\cdot\ln n$ and $P(n) \ll \sqrt{n}$. So we have the estimate

$$\sum_{\substack{n\le x\\ n\in B}} (SDF(n)-P(n))^2 \ll \sum_{n\le x} n\cdot\ln^2 n \ll x^2\cdot\ln^2 x. \qquad (4)$$

Combining (3) and (4) we have

$$\begin{aligned}
\sum_{n\le x} (SDF(n)-P(n))^2 &= \sum_{\substack{n\le x\\ n\in A}} (SDF(n)-P(n))^2+\sum_{\substack{n\le x\\ n\in B}} (SDF(n)-P(n))^2\\
&= \frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1} x}\right),
\end{aligned}$$

where all c_i are computable constants. This proves Theorem 1.

Now we prove Theorem 2. Note that $S(n) - P(n) = 0$, if $n \in A$; and $|S(n) - P(n)| \ll \sqrt{n}$, if $n \in B$. So from the result of the reference [4] and the proving method of Theorem 1 we have

$$\begin{aligned}\sum_{n\le x}(SDF(n)-S(n))^2 &= \sum_{n\le x}(SDF(n)-P(n))^2+\sum_{n\le x}(S(n)-P(n))^2\\ &\quad -2\sum_{n\le x}(S(n)-P(n))\cdot(SDF(n)-S(n))\\ &= \frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right).\end{aligned}$$

This completes the proof of Theorem 2.

References

[1] F. Smarandache, Only Problems, Not Solutions, Chicago, Xiquan Publishing House, 1993.

[2] M. L. Perez, Florentin Smarandache Definitions, Solved and Unsolved Problems, Conjectures and Theorems in Number theory and Geometry, Chicago, Xiquan Publishing House, 2000.

[3] C. Dumitrescu and V. Seleccu, Some Notions and Questions in Number Theory, Erhus University Press, Gelndale, 1994.

[4] Xu Zhefeng, The value distribution property of the Smarandache function, Acta Mathematica Sinica, Chinese Series, **49**(2006), No. 5, 1009-1012.

[5] Jianbin Chen, Value distribution of the F.Smarandache LCM function, Scientia Magna, **3**(2007), No. 2, 15-18.

[6] Jozsef Sandor, On an generalization of the Smarandache function, Notes Numb. Th. Discr. Math, **5**(1999), 41-51.

[7] Jozsef Sandor, On an additive analogue of the function S, Smarandance Notes Numb. Th. Discr. Mathe, **5**(2002), 266-270.

[8] Tom M. Apostol, Introduction to analytical number theory, Spring-Verlag, New York, 1976.

[9] Pan Chengdong and Pan Chengbiao, The elementary proof of the prime theorem, Shanghai Science and Technology Press, Shanghai, 1988.

[10] Zhang Wenpeng, The elementary number theory, Shaanxi Normal University Press, Xi'an, 2007.