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A Variation of Decomposition Under a Length Constraint

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Abstract: Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be graphical properties. A *Smarandachely* $(\mathscr{P}_1, \mathscr{P}_2)$-*decomposition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \cdots, G_l \in \mathscr{P}$ such that $G_i \in \mathscr{P}_1$ or $G_i \notin \mathscr{P}_2$ for integers $1 \leq i \leq l$. Particularly, if $\mathscr{P}_2 = \emptyset$, i.e., a usual *decomposition* of a graph, is a collection of its subgraphs whose union equals the edge set of the graph. In this paper we introduce and initiate a study of a new variation of decomposition namely *equiparity induced path decomposition* of a graph which is defined to be a decomposition in which all the members are induced paths having same parity.

Key Words: Smarandachely $(\mathscr{P}_1, \mathscr{P}_2)$-decomposition, induced path decomposition, equiparity path decomposition, equiparity induced path decomposition.

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§1. Introduction

By a graph $G = (V, E)$ we mean a non-trivial, finite, connected and undirected graph without loops or multiple edges. For terms not defined here, we refer to [3]. Throughout the paper the order and size of G are denoted by n and m respectively.

The origin of the study of graph decomposition and factorization can be seen in various combinatorial problems most of which emerged in the 19th century. Among them the best known are Kirkman's problem of 15 strolling school girls, Dudney's problem of handcuffed prisoners, Euler's problem of 36 army officers , Kirkman's problem of knights and Lucas dancing round problem. However, the earliest works in this direction are not explicitly related to graph decompositions. The first papers (due to J.Peterson, A.B.Kempe, P.G.Tait, P.J.Heawood, D.Konig and others) appeared soon afterwards at the turn of the 19th century. Since that time the interest in graph decompositions has been on increase and real upsurge is witnessed after 1950. Nowadays, graph decomposition problems rank among the most prominent areas of research in graph theory and combinatorics.

As we know a *decomposition* of G is a collection $\psi = \{H_1, H_2, H_3, \ldots, H_k\}$ of subgraphs of G such that every edge of G belongs to exactly one H_i. If each H_i is a path in G, then ψ is called

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a *path decomposition* of G. The minimum cardinality of a path decomposition of G is called the path decomposition number and is denoted by $\pi_a(G)$. The concept of path decomposition was introduced by Harary [4] in the year 1970 and was further studied by Schwenk, Peroche, Stanton, Cowan and James ([5], [7], [9]). Following Harary several variations of decomposition have been introduced and extensively studied by imposing conditions on the members of the decomposition. For instance, unrestricted path cover [5], geodesic path partition [10], simple path cover [2], induced path decomposition [8], equiparity path decomposition [6], graphoidal cover [1] are some variations of decomposition. In this direction we introduce the concept of *equiparity induced path decomposition* and initiate a study of this new decomposition.

§2. Equiparity Induced Path Decomposition

In this section we define the equiparity induced path decomposition and the parameter equiparity induced path decomposition number of a graph G and determine this parameter for some standard graphs such as complete multipartite graphs, wheels, fans, double fans and generalized Petersen graphs. Further we explore the relation between this parameter and some of the existing path decomposition parameters of a given graph G.

Definition 2.1 *An Equiparity induced path decomposition ($\mathcal{ED}$) of a graph G is a path decomposition ψ of G such that the elements of ψ are induced paths having same parity. That is, an $\mathcal{ED}$ is an equiparity as well as induced path decomposition of G. The minimum cardinality of an $\mathcal{ED}$ for a graph G is called the equiparity induced path decomposition number and is denoted by $\pi_{pi}(G)$. Any $\mathcal{ED}$ of G such that $|\psi| = \pi_{pi}(G)$ is called a* minimum equiparity induced path decomposition *of G.*

An equiparity induced path decomposition ψ of a graph is said to be an even parity induced path decomposition ($E\mathcal{ED}$) or an odd parity induced path decomposition ($O\mathcal{ED}$) according as all the paths in ψ are of even length or odd length.

Remark 2.2 Obviously, for any graph G, the edge set $E(G)$ itself is an $\mathcal{ED}$ so that every graph G admits an $\mathcal{ED}$ and hence the parameter π_{pi} is well defined for all graphs.

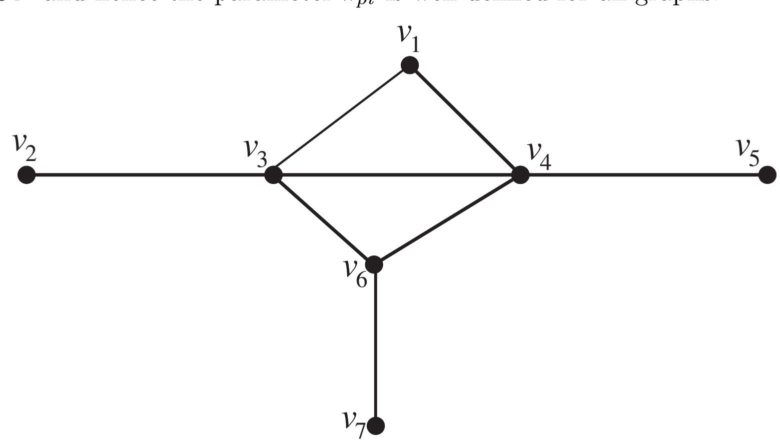

Fig.1

Example 2.3 (i) Consider the graph G given in Fig.1. Let

$$\psi_1 = \{(v_1, v_3, v_6, v_7), (v_2, v_3, v_4, v_5), (v_1, v_4), (v_4, v_6)\}$$
$$\psi_2 = \{(v_2, v_3, v_1), (v_3, v_6, v_7), (v_1, v_4, v_6), (v_3, v_4, v_5)\}$$
$$\psi_3 = \{(v_2, v_3, v_4, v_6), (v_7, v_6, v_3, v_1, v_4, v_5)\}$$
$$\psi_4 = \{(v_1, v_3, v_6, v_7), (v_1, v_4, v_6), (v_2, v_3, v_4, v_5)\}.$$

Then ψ_1 and ψ_2 are $\mathcal{ED}$s of G. Also, all the paths in ψ_1 are of odd length, where as in ψ_2 they are even. But ψ_3 and ψ_4 are not $\mathcal{ED}$s for G, because the former is not induced and the latter is not equiparity. We also note that the minimum cardinality of an $\mathcal{ED}$ for G is 4 and thus both ψ_1 and ψ_2 are minimum $\mathcal{ED}$s of G.

(ii) For paths, the value of π_{pi} is always 1.

(iii) If C_n denotes the cycle on n vertices, then

$$\pi_{pi}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

(iv) Since the edges are the only induced paths in the complete graph K_n on n vertices, we have

$$\pi_{pi}(K_n) = \frac{n(n-1)}{2}.$$

Remark 2.4 If G is a graph of odd size, then it admits only an $O\mathcal{ED}$ and so the value of π_{pi} must be odd. However it is possible for a graph of even size to have both $O\mathcal{ED}$ and $E\mathcal{ED}$; in fact it can permit an $O\mathcal{ED}$ and an $E\mathcal{ED}$ of minimum cardinality as in Example 2.3(i). Also, the value of π_{pi} for a graph with even size can be both even or odd (for example see Theorem 2.9).

To determine the value of π_{pi} for a given graph, the following theorem is useful. If $P = (v_1, v_2, v_3, \ldots, v_n)$ is a path in a graph $G = (V, E)$, the vertices $v_2, v_3, \ldots, v_{n-1}$ are called *internal vertices* of P while v_1 and v_n are called *external vertices* of P.

Theorem 2.5 *For an $\mathcal{ED}$ ψ of a graph G, let $t_\psi = \sum_{p\in\psi} t(P)$ where $t(P)$ denotes the number of internal vertices of the path P and let $t = \max t_\psi$, where the maximum is taken over all $\mathcal{ED}, \psi$ of G. Then $\pi_{pi}(G) = m - t$.*

Proof Let ψ be any $\mathcal{ED}$ of G. Then

$$\begin{aligned} m &= \sum_{p\in\psi} |E(P)| = \sum_{p\in\psi} [t(P)+1] \\ &= \{\sum_{p\in\psi} t(P)\} + |\psi| = t_\psi + |\psi| \end{aligned}$$

Hence $|\psi| = m - t_\psi$ so that $\pi_{pi} = m - t$. □

The following corollaries are the immediate consequences of the above theorem.

Corollary 2.6 *If G is a graph with k vertices of odd degree, then*

$$\pi_{pi}(G) = \frac{k}{2} + \sum_{v\in V(G)} \lfloor \frac{\deg v}{2} \rfloor - t.$$

Corollary 2.7 *For any graph $G, \pi_{pi}(G) \geqslant \frac{k}{2}$. Further, equality holds if and only if there exists an equiparity induced path decomposition ψ of G such that every vertex v of G is an internal vertex of $\lfloor \frac{\deg v}{2} \rfloor$ paths in ψ.*

In the following results, we determine the value of π_{pi} for wheels, complete multipartite graphs, fans, double fans and the generalized Petersen graph.

Theorem 2.8 *If W_n denotes the wheel on n vertices, then*

$$\pi_{pi}(W_n) = \begin{cases} \frac{(n+3)}{2} & \text{when } n \text{ is odd} \\ (n+2) & \text{when } n \text{ is even} \end{cases}$$

Proof If $n = 4$, then $W_4 = K_4$ so that $\pi_{pi}(W_4) = 6$. Now let us assume that $n \geqslant 5$. Let $V(W_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(W_n) = \{v_n v_i : 1 \leqslant i \leqslant n-1\} \cup \{v_i v_{i+1} : 1 \leqslant i \leqslant n-2\} \cup \{v_{n-1}, v_1\}$.

Case 1 n is odd.

Let

$$P_i = (v_i, v_n, v_{i+\frac{n-1}{2}}) \quad \text{for all } i = 1, 2, 3, \ldots, \frac{n-1}{2},$$
$$Q_1 = (v_1, v_2, v_3) \text{ and}$$
$$Q_2 = (v_3, v_4, v_5, \ldots, v_{n-1}, v_1).$$

Then, $\psi = \{P_1, P_2, P_3, \ldots, P_{\frac{n-1}{2}}, Q_1, Q_2\}$ is an $\mathcal{ED}$ of W_n so that $\pi_{pi}(W_n) \leqslant |\psi| = \frac{n-1}{2} + 2 = \frac{n+3}{2}$. Further, any induced path containing the vertex v_n is of length at most two and so the minimum number of induced paths required to decompose the spokes (the edges $v_1v_n, v_2v_n, \ldots, v_{n-1}v_n$) of the wheel is $\frac{n-1}{2}$. Also, since the outer cycle is of even length, we need at least two induced paths to decompose it and hence $\pi_{pi}(W_n) \geq \frac{n-1}{2} + 2 = \frac{n+3}{2}$ so that $\pi_{pi}(W_n) = \frac{n+3}{2}$ when n is odd.

Case 2 n is even.

Let

$$P_i = (v_i, v_n) \quad \text{for all } i = 1, 2, 3, \ldots, n-1,$$
$$Q_1 = (v_1, v_2, v_3, \ldots, v_{n-2}),$$
$$Q_2 = (v_{n-2}, v_{n-1}) \text{ and}$$
$$Q_3 = (v_{n-1}, v_1)$$

Then $\psi = \{P_1, P_2, P_3, \ldots, P_{n-1}, Q_1, Q_2, Q_3\}$ is an $\mathcal{ED}$ so that $\pi_{pi}(W_n) \leqslant |\psi| = n+2$. Moreover, an induced path of W_n cannot contain both an edge of the outer cycle and a spoke. Now, the outer cycle can be decomposed into induced paths of odd length only, because the outer cycle is odd. Therefore, we can have only an $O\mathcal{ED}$ and obviously that will consist of all the $n-1$ spokes together with at least three induced paths of odd length which decompose the outer cycle so that $|\psi| \geqslant n+2$ and this completes the proof of the theorem. □

Theorem 2.9 *If G is a complete k-partite graph $K_{m_1,m_2,m_3,\ldots,m_k}$, with m edges, then*

$$\pi_{pi}(G) = \begin{cases} \dfrac{m}{2} & \text{if } m_i \text{ is odd for at most one } i \\ m & \text{otherwise} \end{cases}$$

Proof Let $(V_1, V_2, \ldots, V_k)$ be the partition of $V(G)$. Obviously the induced paths in G are of length at most two and hence any $\mathcal{ED}$ of G consists of either single edges alone or induced paths of length 2. Moreover the end vertices of the induced paths of length two lie in the same partition. Therefore, when there exist two parts V_i and V_j having odd number of vertices, the edges between V_i and V_j can be decomposed into only single edges and so the edge set $E(G)$ is the only $\mathcal{ED}$ of G in this case. Thus $\pi_{pi}(G) = m$ when there are at least two parts of odd order. On the other hand, when at most one part in $(V_1, V_2, \ldots, V_k)$ is of odd order, the edges between every pair of parts V_i and V_j can be decomposed into induced paths of length two so that $\pi_{pi}(G) \leqslant \frac{m}{2}$. Further, since the length of an induced path in G is at most two we need at least $\frac{m}{2}$ induced paths to decompose G and hence $\pi_{pi}(G) \geqslant \frac{m}{2}$. Thus $\pi_{pi}(G) = \frac{m}{2}$ when at most one part is of odd order. □

Corollary 2.10 *For the complete bipartite graph $K_{r,s}$ we have*

$$\pi_{pi}(K_{r,s}) = \begin{cases} rs & \text{if } rs \text{ is odd} \\ \dfrac{rs}{2} & \text{if } rs \text{ is even} \end{cases}$$

Proof When at most one of the values of r and s is odd, rs is even and it is odd when both r and s are odd. Therefore the result follows by Theorem 2.9. □

For integers s and k with $s \geq 3$ and $0 < k < \frac{s}{2}$, the *generalized Petersen graph* $P(s,k)$ is the simple graph with vertices $\{u_i, v_i : 1 \leq i \leq s\}$ and edges $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k}\}$ where the addition is modulo s.

Theorem 2.11 *For the generalized Petersen graph $P(s,k)$, the value of $\pi_{pi}(P(s,k))$ is $\frac{n}{2}$.*

Proof Obviously the generalized Petersen Graph $P(s,k)$ is a three regular graph of order $2s$ and size $3s$. Therefore by Corollary 2.7 we have $\pi_{pi}(P(s,k) \geqslant s$. Let $P_i = (u_i, u_{i+1}, v_{i+1}, v_{i+1+k})$; $1 \leq i \leq s$, where addition is modulo s. Then P_i is an induced path of length 3 and $\psi = \{P_1, P_2, P_3, \ldots, P_s\}$ is an $\mathcal{ED}$ for $P(s,k)$ so that $\pi_{pi}(P(s,k)) \leqslant |\psi| = s = \frac{m}{3} = \frac{n}{2}$ and hence we obtain the desired result. □

Theorem 2.12 *For the fan $F_n = P_{n-1} + K_1$ with $n > 2$,*

$$\pi_{pi}(F_n) = \begin{cases} n & \text{when } n \text{ is odd} \\ n+1 & \text{when } n \text{ is even} \end{cases}$$

Proof Let $V(F_n) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ where the vertex v_n correspond to K_1 and $E(F_n) = \{v_iv_{i+1} : 1 \leqslant i \leqslant n-2\} \cup \{v_iv_n : i = 1, 2, 3, \ldots, n-1\}$. Since the size of F_n is always odd, any $\mathcal{ED}$ of F_n is an $O\mathcal{ED}$. Now, let

$$Q_i = (v_i, v_n), \quad \textit{for all } i = 1, 2, 3, \ldots, n-1,$$
$$Q_n = (v_1, v_2, v_3, \ldots, v_{n-2}) \text{ and}$$
$$Q_{n+1} = (v_{n-2}, v_{n-1}).$$

Suppose n is odd. Then $\psi_1 = \{Q_1, Q_2, Q_3, \ldots, Q_{n-1}, P_{n-1}\}$ is an $O\mathcal{ED}$ of F_n so that $\pi_{pi}(F_n) \leqslant |\psi_1| = n$. Moreover, the induced paths containing v_n are of length at most two and hence any $O\mathcal{ED}$ of F_n includes all the $n-1$ edges incident at v_n. In addition, we need at least one more path to cover the remaining edges which lie on the path P_{n-1} and hence $\pi_{pi}(F_n) \geqslant n-1+1 = n$. Thus we get $\pi_{pi}(F_n) = n$. If n is even, then $\psi_2 = \{Q_1, Q_2, Q_3, \ldots, Q_{n-1}, Q_n, Q_{n+1}\}$ is an $O\mathcal{ED}$ of F_n. So that $\pi_{pi}(F_n) \leqslant |\psi_2| = n+1$. A similar argument shows that $\pi_{pi}(F_n) = n+1$. □

Theorem 2.13 *For the double fan $G = P_n + (\overline{K_2})$ with $n \geqslant 2$,*

$$\pi_{pi}(G) = \begin{cases} n+1 & \textit{if } n \textit{ is odd} \\ 2n+1 & \textit{if } n \textit{ is even} \end{cases}$$

Proof Let $V(G) = \{u_1, u_2, v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ and $E(G) = \{v_iv_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{u_1v_i : i = 1, 2, 3, \ldots, n\} \cup \{u_2v_i : i = 1, 2, 3, \ldots, n\}$.

Assume that n is odd. Let $Q_i = (u_1, v_i, u_2)$ for all $i = 1, 2, 3, \ldots, n$ and $\psi_1 = \{Q_1, Q_2, Q_3, \ldots, Q_n, P_n\}$. Then ψ_1 is an $E\mathcal{ED}$ of G with cardinality $n+1$ so that $\pi_{pi}(G) \leqslant n+1$. Further, the induced paths of G other than P_n are of length at most two and hence at least n paths of length two are necessary to cover the edges $u_1v_i (i = 1, 2, 3, \ldots, n)$ and $u_2v_i (i = 1, 2, 3, \ldots, n)$. Therefore if ψ is any $\mathcal{ED}$ of G then $|\psi| \geqslant n+1$ and hence we get $\pi_{pi}(G) = n+1$ when n is odd. Now, suppose that n is even. Then size of G is odd which implies that any $\mathcal{ED}$ of G will be an $O\mathcal{ED}$. Since P_n is an odd path in G, P_n along with the remaining edges of G forms an $O\mathcal{ED}$ with cardinality $2n+1$ so that $\pi_{pi}(G) \leqslant 1+2n$. Moreover, P_n is the only odd path in G with length greater than one and hence if ψ is an $O\mathcal{ED}$, it will contain all the edges lying outside P_n. So $|\psi| \geqslant 2n+1$. Thus $\pi_{pi}(G) = 2n+1$ when n is even. □

§3. Bounds for π_{pi}

In this section we obtain some bounds for π_{pi} of a graph in terms of some known graph theoretic parameters. Also we discuss the relation of π_{pi} with some existing decomposition parameters. First we present bounds of π_{pi} in terms of the diameter and the girth of a graph.

Theorem 3.1 *For any graph G with diameter d,*

$$\pi_{pi}(G) \leqslant \begin{cases} m-d+1 & \textit{if } d \textit{ is odd} \\ m-d+2 & \textit{if } d \textit{ is even} \end{cases}$$

Proof Let P be a diameter path(a path whose length is the diameter of the graph) in G. Then P is an induced path of length d. If d is odd, then, the path P together with the remaining edges of G form an $\mathcal{OED}$ of G so that $\pi_{pi}(G) \leqslant m - d + 1$. When d is even, the path $P^{'}$ of length $d-1$ obtained by deleting an edge from P is an odd path and hence G will have an $\mathcal{OED}$ ψ consisting of $P^{'}$ and the remaining edges of G with $|\psi| = 1 + m - (d-1) = m - d + 2$ which gives the desired bound. □

Theorem 3.2 *If G is a graph with girth g, then*

$$\pi_{pi}(G) \leq \begin{cases} m - g + 3 & \text{if } g \text{ is odd} \\ m - g + 4 & \text{if } g \text{ is even} \end{cases}$$

Proof Let C be the shortest cycle in G of length g. Let P be the path obtained from C by deleting a path of length two. Then P is an induced path. By a similar argument followed in Theorem 3.1 the desired result follows. □

Remark 3.3 The bounds given in Theorem 3.1 and Theorem 3.2 are attained for several classes of graphs. For example, it can be easily verified that the complete graphs and complete multipartite graphs in which at most one partition is consisting of an odd number of vertices are such classes of graphs.

As observed in Remark 3.3, one can list several classes of graphs attaining the bounds given in the above theorems; which means the class of those graphs is relatively larger and so the following problems are worth trying.

Problem 3.4 *Characterize the graphs for which*

(i) $\pi_{pi} = m - d + 1$ *when d is odd;*
(ii) $\pi_{pi} = m - d + 2$ *when d is even;*
(iii) $\pi_{pi} = m - g + 3$ *when g is odd;*
(iv) $\pi_{pi} = m - g + 4$ *when g is even.*

Now, it is obvious that the value of π_{pi} of a graph G is ranging from 1 to m where m is the size of G and the lower bound is attained only for paths. On the other hand there are infinitely many graphs attaining the upper bound m. A simple example is a class of complete multipartite graphs as in Theorem 2.9 and the following is another such an infinite family.

Example 3.5 Let G be the graph obtained by pasting two complete graphs at an edge. For example pasting two triangles we get K_4 minus an edge. Now, if $e = (u, v)$ is the edge at which the complete graphs K_r and K_s are pasted, then u and v are adjacent to all the vertices of G. Since the induced paths in G are of length at most two and the edge e does not belong to any induced path of length two the only $\mathcal{ED}$ possible for G is that of the edge set of G and hence we have $\pi_{pi}(G) = m$.

Also it follows from Theorem 3.1 that the diameter necessarily be at most 2 for such graphs. That is, the graphs with $\pi_{pi} = m$ are either complete or of diameter 2. However, the problem of determining these graphs seems to be a little challenging to settle. As a first step we solve the problem in the case of block graphs.

Theorem 3.6 *If G is a block graph which is not a star of odd order, then $\pi_{pi}(G) = m$ if and only if G contains exactly one cut vertex.*

Proof Suppose G is a block graph which is not a star of odd order with $\pi_{pi}(G) = m$ having more than one cut vertex. Let u and v be two cut vertices of G that are adjacent. Then both u and v belong to the same block, say B_k of G. Let $e_u = w_1u$ and $e_v = vw_2$ be two edges of G belonging to two different blocks other than B_k. Then the path $P = (w_1, u, v, w_2)$ is an induced path of length three so that P together with the remaining edges form an $\mathcal{OED}$ of G with cardinality less than m contradicting the assumption that $\pi_{pi}(G) = m$. Hence G contains exactly one cut vertex.

Conversely, suppose G contains exactly one cut vertex, say v. If all the blocks of G are of order 2, then $G = K_{1,s}$ where s is odd and so by Theorem 2.9 we have $\pi_{pi}(G) = m$. If not, let B_r be a block of order greater than 2. Let e be an edge of B_r that is not incident at the cut vertex v. Then e does not belong to any induced path of length greater than one. Moreover, the maximum length of any induced path in G is 2. Hence $E(G)$ is the only $\mathcal{ED}$ for G so that $\pi_{pi}(G) = m$ and this completes the proof. □

Corollary 3.7 *If T is a tree, then $\pi_{pi}(T) = m$ if and only if T is a star of even order.*

Proof Notice that $\pi_{pi} = \dfrac{m}{2}$ for a star of odd order. Therefore the result follows from Theorem 3.6. □

In the following we establish some interesting relations between π_{pi} and some existing path decomposition parameters such as the induced acyclic path decomposition number π_{ia} and equiparity path decomposition number π_p. A decomposition of a graph into induced paths is called an *induced path decomposition* and a decomposition into paths of same parity is called *equiparity path decomposition*. The minimum cardinality of such decompositions are denoted by π_{ia} and π_p respectively.

Theorem 3.8 *For any graph G, we have $\pi_{ia}(G) \leqslant \pi_{pi}(G) \leqslant 2\pi_{ia}(G) - 1$. Further if a and b are two positive integers with $a \leqslant b \leqslant 2a - 1$, then there exists a graph G such that $\pi_{ia}(G) = a$ and $\pi_{pi}(G) = b$.*

Proof The first inequality is immediate because every equiparity induced path decomposition will be an induced acyclic path decomposition. Now, let ψ be an induced acyclic path decomposition of G with r paths of even length and s paths of odd length. If either r or s is zero, then $\pi_{ia}(G) = \pi_{pi}(G)$. Assume that both r and s are positive. Now, split each path of even length in ψ into two paths of odd length and obtain a path decomposition ψ' consisting of these paths of odd length along with all the paths of odd length in ψ. Then ψ' will be an $\mathcal{OED}$ with cardinality $2r + s$ which is obviously at most $2\pi_{ia}(G) - 1$.

Now, let a and b be given integers with $a \leqslant b \leqslant 2a - 1$. We construct a graph G for which $\pi_{ia}(G) = a$ and $\pi_{pi}(G) = b$ as follows. If $a = 1$, then $b = 1$ and so G must be a path.

If $a = 2$ and $b = 2$, let $G = K_{1,4}$ and if $a = 2$ and $b = 3$, let $G = K_{1,3}$. Assume $a \geqslant 3$. Take $b = 2a - 1 - r$ where $0 \leqslant r \leqslant a - 1$. Let G be the graph obtained from the triangle (v_1, v_2, v_3, v_1) by attaching r paths of length 2 along with $2a - 4 - r$ pendant edges at a vertex of the triangle, say v_1. We now prove that $\pi_{ia}(G) = a$ and $\pi_{pi}(G) = b$. Let $x_1', x_2', x_3', \ldots, x_r'$ be the vertices of degree 2 lying outside the triangle and let $x_1, x_2, x_3, \ldots, x_r$ be the pendant vertices adjacent to $x_1', x_2', x_3', \ldots, x_r'$ respectively. Let us denote the remaining pendant vertices of G by $y_1, y_2, y_3, \ldots y_{r-2}, z_1, z_2, z_3, \ldots, z_{2a-2r-2}$ as in Fig.2.

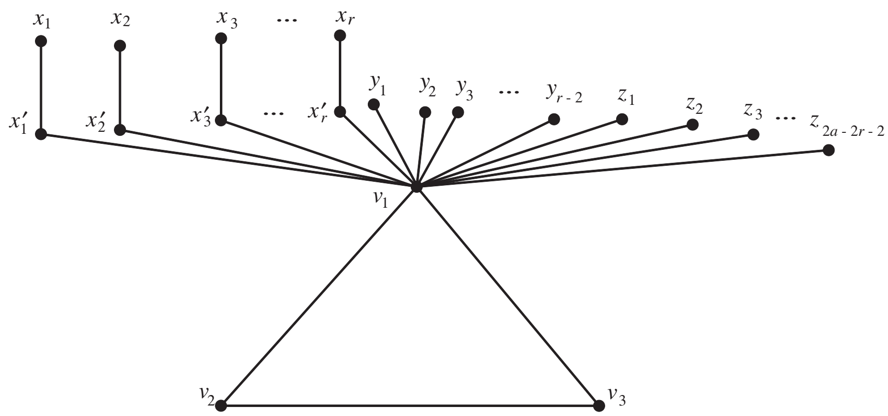

Fig.2

Now let

$$P_1 = (x_1, x_1', v_1, v_2), \quad P_2 = (x_2, x_2', v_1, v_3),$$
$$P_i = (x_i, x_i', v_1, y_{i-2}) \quad \text{for all } i = 3, 4, 5, \ldots, r$$
$$Q_i = (z_i, v_1) \quad \text{for all } i = 1, 2, 3, \ldots, 2a - 2r - 2 \text{ and}$$
$$R_i = (z_{2i-1}, v_1, z_{2i}) \quad \text{for all } i = 1, 2, 3, \ldots, a - r - 1.$$

Then $\psi_1 = \{P_1, P_2, \ldots, P_r, R_1, R_2, \ldots R_{a-r-1}, (v_2, v_3)\}$ is an induced acyclic path decomposition of G with $|\psi_1| = a$ so that $\pi_{ia}(G) \leqslant a$. Further, any induced acyclic path decomposition must contain at least $\dfrac{(2a-2)}{2} = a - 1$ induced paths in order to cover all the $2a - 2$ edges of G incident at the vertex v_1 and also none of these induced paths paths cover the edge v_2v_3 as these paths are induced so that $\pi_{ia}(G) \geqslant a$ and thus $\pi_{ia}(G) = a$.

Next we observe that $\psi_2 = \{P_1, P_2, \ldots, P_r, Q_1, Q_2, \ldots Q_{2a-2r-2}, (v_2, v_3)\}$ is an $\mathcal{OED}$ of G with $|\psi_2| = 2a - 1 - r = b$ so that $\pi_{pi}(G) \leqslant b$. Further let ψ be any $\mathcal{ED}$. Since the edge v_2v_3 cannot be a part of any induced path of length greater than one, it itself must be a member of ψ so that ψ is an $\mathcal{OED}$. Hence among the $2a - 2$ edges incident at v_1, only the edges $x_i'v_1, (i = 1, 2, 3, \ldots, r)$ can be a part of an induced path of of length greater than one and each of the remaining $2a - 2 - r$ edges must be a member of ψ so that $|\psi_2| \geqslant 2a - 2 - r + 1 = b$. Thus $\pi_{pi}(G) = b$ and this completes the proof of the theorem. □

Remark 3.9 Since an equiparity induced path decomposition is an equiparity path decomposition and an equiparity path decomposition is a path decomposition, it follows that

$\pi_a(G) \leqslant \pi_p(G) \leqslant \pi_{pi}(G)$ for any graph G. Further these inequalities can be strict. That is, all the three parameters can be either distinct or all are equal. For example, these parameters coincide in the case of paths, cycles of even length and Petersen graph and if $H = G - v_4v_5$, where G is the graph given in Figure 1, then $\pi_a(H) = 2$, $\pi_p(H) = 3$ and $\pi_{pi}(H) = 5$. The the following interpolation problem naturally arises.

Problem 3.10 *If a, b and c are positive integers with $a \leqslant b \leqslant c$ does there exist a graph G such that $\pi_a(G) = a, \pi_p(G) = b$ and $\pi_{pi}(G) = c$?*

§4. Conclusion and Scope

The theory of decomposition is one of the fastest growing areas of research in graph theory. We have come across varieties of decompositions in the literature and most of them are defined by demanding the members of the decomposition to posses some interesting properties. We have introduced the concept of the equiparity induced path decomposition wherein the concepts of equiparity and induceness have been combined. This study is just a first step in this direction. However, there is wide scope for further research on this parameter and here we list some of them.

(1) *Determine the value of π_{pi} for more classes of graphs like trees, unicyclic graphs and bicyclic graphs.*

(2) *Characterize the graphs for which $\pi_{pi} = \frac{m}{2}$, m, π_{ia}, $2\pi_{ia} - 1$ or $\pi_p = \pi_a$.*

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