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Open Alliance in Graphs

N.Jafari Rad and H.Rezazadeh

(Department of Mathematics of Shahrood University of Technology, Shahrood, Iran)

Email: n.jafarirad@shahroodut.ac.ir, rezazadehadi1363@gmail.com

Abstract: A defensive alliance in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying the condition that for every vertex $v \in S$, the number of v's neighbors is at least as large as the number of v's neighbors in $V - S$. For a subset $T \subset V, T \neq S$, a defensive alliance S is called *Smarandachely T-strong*, if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap ((V - S) \cup T)|$. In this case we say that every vertex in S is *Smarandachely T-strongly defended*. Particularly, if we choose $T = \emptyset$, i.e., a Smarandachely $\emptyset$-strong is called strong defend for simplicity. The boundary of a set S is the set $\partial S = \bigcup_{v \in S} N(v) - S$. An offensive alliance in a graph G is a set of vertices $S \subseteq V$ such that for every vertex v in the boundary of S, the number of v's neighbors in S is at least as large as the number of v's neighbors in $V - S$. In this paper we study open alliance problem in graphs which was posted as an open question in [S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, *Alliances in graphs*, J. Combin. Math. Combin. Comput. 48 (2004) 157-177].

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§1. Introduction

In this paper we study open alliance in graphs. For graph theory terminology and notation, we generally follow [3]. For a vertex v in a graph $G = (V, E)$, the *open neighborhood* of v is the set $N(v) = \{u : uv \in E\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *boundary* of S is the set $\partial S = \bigcup_{v \in S} N(v) - S$. We denote the degree of v in S by $d_S(v) = N(v) \cap S$. The *edge connectivity*, $\lambda(G)$, of a graph G is the minimum number of edges in a set, whose removal results in a disconnected graph. A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$, written $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. For $S \subseteq V$, the *subgraph induced* by S is the graph $G[S] = (S, E \cap S \times S)$.

The study of *defensive alliance* problem in graphs, together with a variety of other kinds of alliances, was introduced in [2]. A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. In this case, we say that every vertex in S is defended from possible attack by vertices in $V - S$. A defensive alliance is called *strong* if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap (V - S)|$. In this case we say that every

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vertex in S is strongly defended. An (strong) alliance S is called *critical* if no proper subset of S is an (strong) alliance. The *defensive alliance number* of G, denoted $a(G)$, is the minimum cardinality of any critical defensive alliance in G. Also the *strong defensive alliance number* of G, denoted $\hat{a}(G)$, is the minimum cardinality of any critical strong defensive alliance in G. For a subset $T \subset V, T \neq S$, a defensive alliance S is called *Smarandachely T-strong*, if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap ((V - S) \cup T)|$. In this case we say that every vertex in S is Smarandachely T-strongly defended. Particularly, if we choose $T = \emptyset$, i.e., a Smarandachely $\emptyset$-strong is called strong defend for simplicity.

The study of *offensive alliances* was initiated by Favaron et al in [1]. A non-empty set of vertices $S \subseteq V$ is called an *offensive alliance* if for every $v \in \partial(S)$, $|N(v) \cap S| \geq |N[v] \cap (V - S)|$. In this case we say that every vertex in $\partial(S)$ is *vulnerable* to possible attack by vertices in S. An offensive alliance is called *strong* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| > |N[v] \cap (V - S)|$. In this case we say that every vertex $\partial(S)$ is *very vulnerable*. The *offensive alliance number*, $a_o(G)$ of G, is the minimum cardinality of any critical offensive alliance in G. Also the *strong offensive alliance number*, $\hat{a}_o(G)$ of G, is the minimum cardinality of any critical strong offensive alliance in G.

In [2] the authors left the study of open alliances as an open question. In this paper we study open alliance in graphs. An alliance is called *open* (or *total*) if it is defined completely in terms of open neighborhoods. We study open defensive alliances as well as open offensive alliances in graphs.

Recall that a vertex of degree one in a graph G is called a *leaf* and its neighbor is a *support vertex*. Let $S(G)$ denote the set all support vertexes of a graph G.

§2. Open Defensive Alliance

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is an *open defensive alliance* if for every vertex $v \in S$, $|N(v) \cap S| \geq |N(v) \cap (V - S)|$. A set $S \subseteq V$ is an *open strong defensive alliance* if for every vertex $v \in S$, $|N(v) \cap S| > |N(v) \cap (V - S)|$. An open (strong) defensive alliance S is called *critical* if no proper subset of S is an open (strong) defensive alliance. The *open defensive alliance number*, $a_t(G)$ of G, is the minimum cardinality of any critical open defensive alliance in G, and the *strong open defensive alliance number*, $\hat{a}_t(G)$ of G, is the minimum cardinality of any critical open strong defensive alliance in G.

We remark that with this definition, strong defensive alliance is equivalent to open defensive alliance, and so we have the following observation.

Observation 2.1 For any graph G, $a_t(G) = \hat{a}(G)$.

Thus we focus on open strong defensive alliances in G. We refer to an $\hat{a}_t(G)$-set as a minimum open strong defensive alliance in G. By definition we have the following.

Observation 2.2 For any $\hat{a}_t(G)$-set S in a graph G, $G[S]$ is connected.

Observation 2.3 Let S be an $\hat{a}_t(G)$-set in a graph G, and $v \in S$. If $deg_{G[S]}(v) = 1$, then

$deg_G(v) = 1$.

Note that for any graph G of n vertices $2 \leq \hat{a}_t(G) \leq n$. In the following we characterize all graphs of order n having open strong defensive alliance number n. For an integer n let $\mathcal{E}_n$ be the class of all graphs G such that $G \in \mathcal{E}_n$ if and only if one of the following holds:
(1) G is a path on n vertices, (2) G is a cycle on n vertices, (3) G is obtained from a cycle on n vertices by identifying two non adjacent vertices.

Theorem 2.4 *For a connected graph G of n vertices, $\hat{a}_t(G) = n$ if and only if $G \in \mathcal{E}_n$.*

Proof First we show that $\hat{a}_t(P_n) = \hat{a}_t(C_n) = n$. Suppose to the contrary, that $\hat{a}_t(P_n) < n$. Let S be a $\hat{a}_t(P_n)$-set. By Observation 2.2, $G[S]$ is connected. So $G[S]$ is a path. Let $v \in S$ be a vertex such that $deg_{G[S]}(v) = 1$. By Observation 2.3, $deg_G(v) = 1$. Then $G[S] = P_n$, a contradiction. Thus $\hat{a}_t(P_n) = n$. Similarly, for any other graph in $\mathcal{E}_n$, $\hat{a}_t(G) = n$.

For the converse suppose that G is a graph of n vertices and $\hat{a}(G) = n$. If $\Delta(G) \leq 2$, then G is a path or a cycle on n vertices, as desired. Suppose that $\Delta(G) \geq 3$. Let v be a vertex of maximum degree in G. Since $V(G) \setminus \{v\}$ is not an open strong defensive alliance in G, there is a vertex $v_1 \in N(v)$ such that $deg(v_1) \leq 2$. If $deg(v_1) = 1$, then $V(G) \setminus \{v_1\}$ is an open strong defensive alliance, which is a contradiction. So $deg(v_1) = 2$. Since $V(G) \setminus \{v_1\}$ is not an open strong defensive alliance, there is a vertex $v_2 \in N(v_1)$ such that $deg(v_2) \leq 2$. If $deg(v_2) = 1$, then $V(G) \setminus \{v_2\}$ is an open strong defensive alliance, which is a contradiction. So $deg(v_2) = 2$. Since $V(G) \setminus \{v_1, v_2\}$ is not an open strong defensive alliance, there is a vertex $v_3 \in N(v_2)$ such that $deg(v_3) \leq 2$. Continuing this process we obtain a path $v_1 - v_2 - ... - v_k$ for some k such that $deg(v_i) = 2$ for $1 \leq i < k$ and either $deg(v_k) = 1$ or $v_k = v$. If $deg(v_k) = 1$, then $V(G) \setminus \{v_1, ..., v_k\}$ is an open strong defensive alliance for G. This is a contradiction. So $v_k = v$. If $deg(v) \geq 5$, then $V(G) \setminus \{v_1, v_2, ..., v_{k-1}\}$ is an open strong defensive alliance for G, a contradiction. So $deg(v) = \Delta(G) = 4$. Since $V(G) \setminus \{v_1, v_2, ..., v_k\}$ is not an open strong defensive alliance, there is a vertex $w_1 \in N(v) \setminus \{v_1, v_{k-1}\}$ with $deg(w_1) \leq 2$. If $deg(w_1) = 1$ then $V(G) \setminus \{w_1\}$ is an open defensive alliance, a contradiction. So $deg(w_1) = 2$. Since $V(G) \setminus \{v_1, v_2, ..., v_k, w_1\}$ is not an open strong defensive alliance, there is a vertex $w_2 \in N(w_1)$ such that $deg(w_2) = 2$. As before, continuing the process, we deduce that there is a path $w_1 - w_2 - ... - w_l$ for some l such that $deg(v_i) = 2$ for $1 \leq i < l$ and $v_l = v$. Since $\Delta(G) = 4$, we conclude that G is obtained by identifying a vertex of C_k with a vertex of C_l. This completes the result. □

As a consequence we have the following result.

Corollary 2.5 *For a connected graph G, $\hat{a}_t(G) = 2$ if and only if $G = P_2$.*

For a nonempty set S in a graph G and a vertex $x \in S$, we let $deg_S(v) = N(v) \cap S$. So a set $S \subseteq V$ is an open defensive alliance if for every vertex $v \in S$, $deg_S(v) \geq deg_{V-S}(v) + 1$. Notice that this is equivalent to $2deg_S(v) \geq deg(v) + 1$.

Proposition 2.6 *For any graph G, $\hat{a}_t(G) = 3$, if and only if $\hat{a}_t(G) \neq 2$, and G has an induced subgraph isomorphic to either (1) the path $P_3 = u - v - w$, where $deg(u) = deg(w) = 1$ and $2 \leq deg(v) \leq 3$, or (2) the cycle C_3, where each vertex is of degree at most three.*

Proof Let G be a graph. Suppose that $\hat{a}_t(G) \neq 2$. If G has an induced subgraph $P_3 = u - v - w$, where $deg(u) = deg(w) = 1$ and $2 \leq deg(v) \leq 3$, then $\{u, v, w\}$ is an open strong defensive alliance, and so $\hat{a}_t(G) = 3$. Similarly, if (2) holds, we obtain $\hat{a}_t(G) = 3$.

Conversely, suppose that $\hat{a}_t(G) = 3$. So $\hat{a}_t(G) \neq 2$. Let $S = \{u, v, w\}$ be a $\hat{a}_t(G)$-set. By Observation 2.2, $G[S]$ is connected. If $G[S]$ is a path, then we let $deg_{G[S]}(u) = deg_{G[S]}(w) = 1$. By definition $deg_G(u) = deg_G(w) = 1$. If $deg_G(v) \geq 4$, then S is not an open strong defensive alliance, which is a contradiction. So $2 \leq deg_G(v) \leq 3$. It remains to suppose that $G[S]$ is a cycle. If a vertex of S has degree at least four in G, then S is not an open strong defensive alliance, a contradiction. Thus any vertex of S has degree at most three in G. □

Let G_1 be a graph obtained from K_4 by removing two edge such that the resulting graph G has a pendant vertex. Let G_2 be a graph obtained from K_4 by removing an edge, with vertices $\{v_1, v_2, v_3, v_4\}$, where $deg(v_1) = deg(v_2) = 2$.

Proposition 2.7 *For any graph G, $\hat{a}_t(G) = 4$ if and only if $\hat{a}_t(G) \notin \{2, 3\}$, and G has an induced subgraph isomorphic to one of the following:*

(1) P_4, with vertices, in order, v_1, v_2, v_3 and v_4, where $deg(v_1) = deg(v_4) = 1$, and $deg(v_2)$ and $deg(v_3)$ are at most three;

(2) C_4, where each vertex is of degree at most three;

(3) K_4, where each vertex has degree at most five;

(4) $K_{1,3}$, with vertices $\{v_1, v_2, v_3, v_4\}$, where $deg(v_i) = 1$ for $i = 2, 3, 4$, and $deg(v_1) \leq 5$;

(5) G_1, where $deg(v_i) \leq 5$ for $i = 1, 2, 3, 4$;

(6) G_2, where $deg(v_i) \leq 3$ for $i = 1, 2$, and $deg(v_i) \leq 5$ for $i = 3, 4$.

Proof It is a routine matter to see that if $\hat{a}_t(G) \notin \{2, 3\}$, and G has an induced subgraph isomorphic to (i) for some $i \in \{1, 2, ..., 6\}$, then $\hat{a}_t(G) = 4$. Suppose that $\hat{a}_t(G) = 4$. Let $S = \{v_1, v_2, v_3, v_4\}$ be a $\hat{a}_t(G)$-set. By Observation 2.2 $G[S]$ is connected. If $G[S]$ is a path, then we assume that $deg_{G[S]}(v_i) = 1$ for $i = 1, 4$, and $deg_{G[S]}(v_i) = 2$ for $i = 2, 3$. Now by Observation 2.3 $deg(v_i) = 1$ for $i = 1, 4$, and $4 = 2deg_{G[S]}(v_i) \geq deg(v_i) + 1$ which implies that $deg(v_i) \leq 3$ for $i = 2, 3$. We deduce that G has an induced subgraph isomorphic to (1). So suppose that $G[S]$ is not a path. If $G[S]$ is a cycle then $4 = 2deg_{G[S]}(v_i) \geq deg(v_i) + 1$ which implies that $deg(v_i) \leq 3$ for $i = 1, 2, 3, 4$, and so G has an induced subgraph isomorphic to (2). We assume now that $\Delta(G[S]) > 2$. So $\Delta(G[S]) = 3$. Let $deg_{G[S]}(v_1) = 3$. If any vertex of $G[S]$ is of maximum degree then $6 = 2deg_{G[S]}(v_i) \geq deg(v_i) + 1$ which implies that $deg(v_i) \leq 5$ for $i = 1, 2, 3, 4$. So G has an induced subgraph isomorphic to (3). Thus we suppose that $G[S]$ is not complete graph. If $deg_{G[S]}(v_i) = 1$ for $i = 2, 3, 4$, then by Observation 2.3 $deg(v_i) = 1$ for $i = 2, 3, 4$, and $6 = 2deg_{G[S]}(v_1) \geq deg(v_1) + 1$, which implies that $deg(v_1) \leq 5$. In this case G has an induced subgraph isomorphic to (4). The other possibilities are similarly verified. □

Proposition 2.8 *For the complete graph K_n , $\hat{a}_t(K_n) = \lceil \frac{n}{2} \rceil + 1$.*

Proof Let S be a $\hat{a}_t(K_n)$-set and let $v \in S$. It follows that $|N(v) \cap S| \geq \lceil \frac{n}{2} \rceil$. So

$|S| \geq \lceil \frac{n}{2} \rceil + 1$. On the other hand let S be any subset of $\lceil \frac{n}{2} \rceil + 1$ vertices of K_n. For any vertex $v \in S$, $\frac{deg(v)-1}{2} \geq \lfloor \frac{n}{2} \rfloor - 1 \geq deg_{V-S}(v)$. Since $deg(v) = deg_S(v) + deg_{V-S}(v)$, $deg_S(v) - 1 \geq deg_{V-S}(v)$. This means that S is a critical open strong defensive alliance, and the result follows. □

Proposition 2.9 $\hat{a}_t(K_{r,s}) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 2$.

Proof Let V_r and V_s be the partite sets of $K_{r,s}$ with $|V_r| = r$ and $|V_s| = s$. Let $S = S_r \cup S_s$ be a $\hat{a}_t(K_{r,s})$-set, where $S_i \subseteq V_i$ for $i = r, s$. For $i \in \{r, s\}$ and a vertex $v \in S_i$, $deg_S(v) \geq \lfloor \frac{n-i}{2} \rfloor$, where $n = r + s$. This implies that $|S| \geq \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 2$. On the other hand any set consisting $\lfloor \frac{r}{2} \rfloor + 1$ vertices in V_r and $\lfloor \frac{s}{2} \rfloor + 1$ vertices in V_s forms an open strong defensive alliance. This completes the proof. □

Similarly the following is verified.

Proposition 2.10

(1) $\hat{a}_t(W_n) = \lceil \frac{n+1}{2} \rceil + 1$;

(2) $\hat{a}_t(P_m \times P_n) = max\{m, n\}$ *if* $min\{m, n\} = 1$, *and* $\hat{a}_t(P_m \times P_n) = min\{m, n\}$ *if* $min\{m, n\} \geq 2$.

Proposition 2.11 *If every vertex of a graph G has odd degree then $a_t(G) = \hat{a}_t(G)$.*

Proof Let G be a graph and every vertex of G has odd degree. First it is obvious that $a_t(G) = \hat{a}(G) \leq \hat{a}_t(G)$. Let S be a $a_t(G)$-set and $v \in S$. By definition $deg_S(v) \geq deg_{V-s}(v)$. Since v is of odd degree, we obtain $deg_S(v) \geq deg_{V-s}(v) + 1$. This means that S is an open strong defensive alliance in G, and so $\hat{a}_t(G) \leq a_t(G)$. □

So if every vertex of a graph G has odd degree then any bound of $a_t(G)$ holds for $\hat{a}_t(G)$. We next obtain some bounds for the open defensive alliance number of a graph G.

Proposition 2.12 *For a connected graph G of order n, $\hat{a}_t(G) \leq n - \left\lfloor \frac{\delta(G)-1}{2} \right\rfloor$.*

Proof Let v be a vertex of minimum degree in a connected graph G. Consider a subset $S \subseteq N[v]$ with $|S| = \lfloor \frac{\delta(G)-1}{2} \rfloor$. It follows that $V(G) \setminus S$ is a critical open strong alliance. □

Proposition 2.13 *For any graph G, $\hat{a}_t(G) \geq \lceil \frac{\delta(G)+3}{2} \rceil$.*

Proof Let S be a $\hat{a}_t(G)$-set in a graph G, and let $v \in S$. By definition $deg_S(v) - 1 \geq deg_{V-S}(v)$. By adding $deg_{V-S}(v)$ to both sides of this inequality we obtain $deg_{V-S}(v) - 1 \leq \frac{deg(v)-1}{2}$. By adding $deg_S(v)$ to both sides of this inequality we obtain $\frac{deg(v)+1}{2} \leq deg_S(v)$. But $deg_S(v) \leq |S| - 1$ and $\delta(G) \leq deg(v)$. We deduce that $\frac{\delta(G)+3}{2} \leq |S|$. □

Proposition 2.14 *For any graph G, $a(G) \leq \hat{a}_t(G) - 1$.*

Proof Let S be a $\hat{a}_t(G)$-set in a graph G, and $w \in S$. Let $S' = S - \{w\}$, and $v \in S'$. It follows that $deg_{S'}(v) = deg_S(v) - deg_{\{w\}}(v) \geq deg_{V-S}(v) + 1 - deg_{\{w\}}(v) = deg_{V-S'}(v) + 1 - 2deg_{\{w\}}(v) \geq deg_{V'_S}(v)$, as desired. □

Let $\Pi = [V_1, V_2]$ be a partition of the vertices of a graph G such that there are $\lambda(G)$ edges between V_1 and V_2. Π is called *singular* $\lambda-$*bipartite* if $min\{|V_1|, |V_2|\} = 1$, and *non*$-$*singular* $\lambda -$ *bipartite* if $min\{|V_1|, |V_2|\} > 1$.

Proposition 2.15 *Let G be a graph such that every vertex of G has odd degree. If $\lambda(G) < \delta(G)$ then $\hat{a}_t(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Proof Let $\Pi = [V_1, V_2]$ be a partition of the vertices of a graph G such that there are $\lambda(G)$ edges between V_1 and V_2. Without loss of generality assume that $|V_1| < |V_2|$. This implies that $|V_1| \leq \lfloor \frac{n}{2} \rfloor$. Since $\lambda(G) < \delta(G)$, we have $|V_i| \geq 2$ for $i = 1, 2$. As a result Π is non-singular λ-bipartite. If V_1 is not an open defensive alliance then there is a vertex $u \in V_1$ such that $|N(u) \cap V_1| < |N(u) \cap V_2|$. Then $\Pi_1 = [V_1 - \{u\}, V_2 \cup \{u\}]$ is a partition of the vertices of G and there are less than $\lambda(G)$ edges between $V_1 - \{u\}$ and $V_2 \cup \{u\}$. But $|\Pi_1| = |\Pi| - deg_{V_2}(u) + deg_{V_1}(u)$. So $|\Pi_1| < |\Pi|$. This contradicts the assumption $|\Pi| = \lambda(G)$. Thus V_1 is an open defensive alliance in G and the result follows. □

§3. Open Offensive Alliance

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is an *open offensive alliance* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| \geq |N(v) \cap (V - S)|$. In other words a set $S \subseteq V$ is an open offensive alliance if for every vertex $v \in \partial(S)$, $deg_S(v) \geq deg_{V-S}(v)$, and this is equivalent to $deg(v) \geq 2deg_{V-S}(v)$. A set $S \subseteq V$ is an *open strong offensive alliance* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| > |N(v) \cap (V - S)|$ or, equivalently, $d_S(v) > d_{V-S}(v)$, where $d_S(v) = N(v) \cap S$. An open (strong) offensive alliance S is called *critical* if no proper subset of S is an open (strong) offensive alliance. The *open offensive alliance number*, $a_{ot}(G)$ of G, is the minimum cardinality of any critical open offensive alliance in G, and the *strong open offensive alliance number*, $\hat{a}_{ot}(G)$ of G, is the minimum cardinality of any critical open strong offensive alliance in G.

If S is a critical open offensive alliance of a graph G and $|S| = a_{ot}(G)$, then we say that S is an $a_{ot} - set$ of G. The next proposition follows from the definitions.

Proposition 3.1 *For all graphs G, $a_o(G) = \hat{a}_{ot}(G)$ and $a_{ot}(G) \leq \hat{a}_{ot}(G)$.*

Thus we focus on open offensive alliances in G.

Theorem 3.2 *For a graph G of order n with $\Delta(G) \leq 2$, $a_{ot}(G) = 1$.*

Proof Suppose $S = \{v\}$, where $deg(v) = \triangle(G) \leq 2$. Since for every $w \in \partial S$, $deg_S(w) = 1$ and $deg_{V-S}(w) \leq 1$. Therefore, $d_S(w) \geq d_{V-S}(w)$. So the result immediately follows. □

Corollary 3.3 *For any cycle C_n and path P_n, $a_{to}(C_n) = a_{to}(P_n) = 1$.*

The following has a straightforward proof and therefore we omit its proof.

Proposition 3.4

(1) $a_{ot}(K_n) = \lfloor \frac{n}{2} \rfloor$*;*

(2) For $1 \leq m \leq n$*,* $a_{ot}(K_{m,n}) = \lceil \frac{m}{2} \rceil$*;*

(3) For any wheel W_n *with* $n \neq 4$*,* $a_{ot}(W_n) = \lceil \frac{n}{3} \rceil + 1$*;*

(4) If every vertex of a graph G *has odd degree then* $a_{ot}(G) = a_o(G)$*.*

We next obtain some bounds for the open offensive alliance number of a graph G.

Proposition 3.5 *For all graphs* G*,* $a_{to}(G) \geq \lfloor \frac{\delta(G)}{2} \rfloor$*.*

Proof Let S be a $a_{ot} - set$ and $v \in \partial S$. By definition for any vertex v of ∂S, $d_S(v) \geq d_{V-S}(v)$. By adding $d_S(v)$ to both sides of this inequality we obtain $d_S(v) \geq \frac{\delta(v)}{2}$. Also it is clear that $a_{to}(G) \geq d_S(v)$ and $\delta(v) \geq \delta$. This completes the proof. □

Let $\alpha(G)$ denote the *vertex covering number* of G. That is the minimum cardinality of a subset S of vertices of G that contains at least one endpoint of every edge.

Proposition 3.6 *For all graphs* G*,*

(1) $a_{to}(G) \leq \lfloor \frac{n}{2} \rfloor$*;*

(2) $a_{to}(G) \leq \alpha(G)$*.*

Proof (1) Let $f : V \longrightarrow \{a, b\}$ be a vertex coloring of G such that the number of edges whose end vertices have the same color is minimum. Let $O = \{uv : f(u) = f(v)\}$, $A = \{u : f(u) = a\}$ and $B = \{u : f(u) = b\}$. Without loss of generality assume that $|B| \leq |A|$. Suppose that B is not an open offensive alliance in G. So three is a vertex $v \in A$ such that $deg_B(v) < deg_A(v)$. Let $f' : V \longrightarrow \{a, b\}$ be a vertex coloring of G with $f'(v) \neq f(v)$ and $f'(x) = f(x)$ if $x \neq v$. Let $O' = \{uv : f'(u) = f'(v)\}$, $A' = A - \{v\}$ and $B' = B \cup \{v\}$. Then $|O'| = |O| - deg_A(v) + deg_B(v)$. But $deg_B(v) < deg_A(v)$. We deduce that $|O'| < |O|$. This is a contradiction since $|O|$ is minimum. Thus B is an open offensive alliance in G, and so the result follows.
(2) Let S be a $\alpha(G)$-set and let $v \in \partial(S)$. Since S is a vertex covering, $deg_S(v) \geq deg_{V-S}(v) + 1 \geq deg_{V-S}(v)$. This implies that S is an open offensive alliance, and the result follows. □

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