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Open Distance-Pattern Uniform Graphs

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Abstract: Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, each vertex u in G is associated with the set $f^o_M(u) = \{d(u, v) \,:\, v \in M,\ u \neq v\}$, called its open M-distance-pattern. A graph G is called a *Smarandachely uniform k-graph* if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f^o_{M_i}(u) = f^o_{M_j}(u)$ and $f^o_{M_i}(u) = f^o_{M_j}(v)$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a *k-family of open distance-pattern uniform (odpu-) set of G* and the minimum cardinality of odpu-sets in G, if they exist, is called the *Smarandachely odpu-number* of G, denoted by $od^S_k(G)$. Usually, a Smarandachely uniform 1-graph G is called an *open distance-pattern uniform (odpu-) graph*. In this case, its odpu-number $od^S_k(G)$ of G is abbreviated to $od(G)$. In this paper we present several fundamental results on odpu-graphs and odpu-number of a graph.

Key Words: Smarandachely uniform k-graph, open distance-pattern, open distance-pattern, uniform graphs, open distance-pattern uniform (odpu-) set, Smarandachely odpu-number, odpu-number.

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§1. Introduction

All graphs considered in this paper are finite, simple, undirected and connected. For graph theoretic terminology we refer to Harary [6].

The concept of open distance-pattern and open distance-pattern uniform graphs were suggested by B.D. Acharya. Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, the open M-distance-pattern of a vertex u in G is defined to be the set $f^o_M(u) = \{d(u, v) : v \in M,\ u \neq v\}$, where $d(x, y)$ denotes the distance between the vertices x and y in G. A graph G is called a *Smarandachely uniform k-graph* if there exist subsets $M_1, M_2, \cdots, M_k$ for an integer $k \geq 1$ such that $f^o_{M_i}(u) = f^o_{M_j}(u)$ and $f^o_{M_i}(u) = f^o_{M_j}(v)$ for $1 \leq i, j \leq k$ and $\forall u, v \in V(G)$. Such subsets $M_1, M_2, \cdots, M_k$ are called a *k-family of open distance-pattern uniform (odpu-) set of G* and the minimum cardinality of odpu-sets in G, if they exist, is called the *Smarandachely odpu-number* of G, denoted by $od^S_k(G)$. Usually, a Smarandachely uniform 1-graph G is called an *open distance-pattern uniform (odpu-) graph*. In this case, its odpu-number $od^S_k(G)$ of G is abbreviated to $od(G)$. We need the following theorem.

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Theorem 1.1([5]) *Let G be a graph of order $n, n \geq 4$. Then the following conditions are equivalent.*

(i) The graph G is self-centred with radius $r \geq 2$ and for every $u \in V(G)$, there exists exactly one vertex v such that $d(u, v) = r$.

(ii) The graph G is r-decreasing.

(iii) There exists a decomposition of $V(G)$ into pairs $\{u, v\}$ such that $d(u, v) = r(G) > max(d(u, x), d(x, v))$ for every $x \in V(G) - \{u, v\}$.

In this paper we present several fundamental results on odpu-graphs and odpu-number of a graph G.

§2. Odpu-Sets in Graphs

It is clear that an odpu-set in any nontrivial graph must have at least two vertices. The following theorem gives a basic property of odpu-sets.

Theorem 2.1 *In any graph G, if there exists an odpu-set M, then $M \subseteq Z(G)$ where $Z(G)$ is the center of the graph G. Also $M \subseteq Z(G)$ is an odpu-set if and only if $f_M^o(v) = \{1, 2, \ldots, r(G)\}$, for all $v \in V(G)$.*

proof Let G have an odpu-set $M \subseteq V(G)$ and let $v \in M$. Suppose $v \notin Z(G)$. Then $e(v) > r(G)$. Hence there exists a vertex $u \in V(G)$ such that $d(u, v) > r(G)$. Since $v \in M$, $f_M^o(u)$ contains an element, which is greater than $r(G)$. Now let $w \in V(G)$ be such that $e(w) = r(G)$. Then $d(w, v) \leq r(G)$ for all $v \in M$. Hence $f_M^o(w)$ does not contain an element greater than $r(G)$, so that $f_M^o(u) \neq f_M^o(w)$. Thus M is not an odpu-set, which is a contradiction. Hence $M \subseteq Z(G)$.

Now, let $M \subseteq Z(G)$ be an odpu-set. Then $max \quad f_M^o(v) = r(G)$. Let $u \in M$ be such that $d(u, v) = r(G)$. Let the shortest $u - v$ path be $(u = v_1, v_2, \cdots, v_{r(G)} = v)$. Then v_1 is adjacent to u. Therefore, $1 \in f_M^o(v_1)$. Since M is an odpu-set, $1 \in f_M^o(x)$ for all $x \in V(G)$. Now, $d(v_2, u) = 2$, whence $2 \in f_M^o(v_2)$. Since M is an odpu-set, $2 \in f_M^o(x)$ for all $x \in V(G)$. Proceeding like this, we get $\{1, 2, 3, \cdots, r(G)\} \subseteq f_M^o(x)$ and since $M \subseteq Z(G)$, $f_M^o(x) = \{1, 2, 3, \cdots, r(G)\}$ for all $x \in V$. The converse is obvious. □

Corollary 2.2 *A connected graph G is an odpu-graph if and only if the center $Z(G)$ of G is an odpu-set.*

Proof Let G be an odpu-graph with an odpu-set M. Then $f_M^o(v) = \{1, 2, \ldots, r(G)\}$ for all $v \in V(G)$. Since $f_{Z(G)}^o(v) \supseteq f_M^o(v)$ and $d(u, v) \leq r(G)$ for every $v \in V$ and $u \in Z(G)$, it follows that $Z(G)$ is an odpu set of G. The converse is obvious. □

Corollary 2.3 *Every self-centered graph is an odpu-graph.*

Proof Let G be a self-centered graph. Take $M = V(G)$. Since G is self-centered, $e(v) = r(G)$ for all $v \in V(G)$. Therefore, $f_M^o(v) = \{1, 2, \cdots, r(G)\}$ for all $v \in V(G)$, so that M is an odpu-set for G. □

Remark 2.4 The converse of Corollary 2.3 is not true. For example the graph $K_2 + \overline{K_2}$, is not self-centered but it is an odpu-graph. Moreover, there exist self-centered graphs having a proper subset of $Z(G) = V(G)$ as an odpu-set.

Theorem 2.5 *If G is an odpu-graph with $n \geq 3$, then $\delta(G) \geq 2$ and G is 2-connected.*

Proof Let G be an odpu-graph with $n \geq 3$ and let M be an odpu-set of G. If G has a pendant vertex v, it follows from Theorem 2.1 that $v \notin M$. Also, v is adjacent to exactly one vertex $w \in V(G)$. Since M is an odpu-set, $max\ \ f_M^o(w) = r(G)$. Therefore, there exists $u \in M$ such that $d(u, w) = r(G)$. Now $d(u, v) = r(G) + 1$ and $f_M^o(v)$ contains $r(G) + 1$. Hence $f_M^o(v) \neq f_M^o(w)$, a contradiction. Thus $\delta(G) \geq 2$.

Now suppose G is not 2-connected. Let B_1 and B_2 be blocks in G such that $V(B_1) \cap V(B_2) = \{u\}$. Since, the center of a graph lies in a block, we may assume that the center $Z(G) \subseteq B_1$. Let $v \in B_2$ be such that $uv \in E(G)$. Then there exists a vertex $w \in M$ such that $d(u, w) = r(G)$ and $d(v, w) = r(G) + 1$, so that $r(G) + 1 \in f_M^o(u)$, which is a contradiction. Hence G is 2-connected. □

Corollary 2.6 *A tree T has an odpu-set M if and only if T is isomorphic to P_2.*

Corollary 2.7 *If G is a unicyclic odpu-graph, then G is isomorphic to a cycle.*

Corollary 2.8 *A block graph G is an odpu-graph if and only if G is complete.*

Corollary 2.9 *In any graph G, if there exists an odpu-set M, then every subset M' of $Z(G)$ such that $M \subseteq M'$ is also an odpu-set.*

Thus Corollary 2.9 shows that in a limited sense the property of subsets of $V(G)$ being odpu-sets is *super-hereditary* within $Z(G)$. The next remark gives an algorithm to recognize odpu-graphs.

Remark 2.10 Let G be a finite simple connected graph. The the following algorithm recognizes odpu-graphs.

Step-1: Determine the center of the graph G.

Step-2: Generate the $c \times n$ distance matrix $D(G)$ of G where $c = |Z(G)|$.

Step-3: Check whether each column C_i has the elements $1, 2, \ldots, r$.

Step-4: If then, G is an odpu-graph.

Or else G is not an odpu-graph.

The above algorithm is efficient since we have polynomial time algorithm to determine $Z(G)$ and to compute the matrix $D(G)$.

Theorem 2.11 *Every odpu-graph G satisfies, $r(G) \leq d(G) \leq r(G)+1$. Further for any positive integer r, there exists an odpu-graph with $r(G) = r$ and $d(G) = r+1$.*

Proof Let G be an odpu-graph. Since $r(G) \leq d(G)$ for any graph G, it is enough to prove that $d(G) \leq r(G) + 1$. If G is a self-centered graph, then $r(G) = d(G)$. Assume G is not self-centered and let u and v be two antipodal vertices of G. Since G is an odpu-graph, $Z(G)$ is an odpu-set and hence there exist vertices $u', v' \in Z(G)$ such that $d(u, u') = 1$ and $d(v, v') = 1$. Now, G is not self-centered, and $d(u, v) = d$, implies $u, v \notin Z(G)$. If $d > r + 1$; since $d(u, u') = d(v, v') = 1$, the only possibility is $d(u', v') = r$, which implies $d(u, v') = r + 1$. But $v' \in Z(G)$ and hence $r + 1 \in f^o_M(u)$, which is not possible. Hence $d(u, v) = d \leqslant r + 1$ and the result follows.

Now, let r be any positive integer. For $r = 1$ take $G = K_2 + \bar{K_n}, n \geqslant 2$. For $r \geqslant 2$, let G be the graph obtained from C_{2r} by adding a vertex v_e corresponding to each edge e in C_{2r} and joining v_e to the end vertices of e. Then, it is easy to check that an odpu-set of the resulting graph is $V(C_{2r})$. □

However, it should be noted that $d = r+1$ is not a sufficient condition for the graph to be an odpu-graph. For the graph G consisting of the cycle C_r with exactly one pendent edge at one of its vertices, $d = r + 1$ but G is not an odpu-graph.

Remark 2.12 Theorem 2.11 states that there are only two classes of odpu-graphs, those which are self-centered or those for which $d(G) = r(G) + 1$. Hence, the problem of characterizing odpu-graphs reduces to the problem of characterizing odpu-graphs with $d(G) = r(G) + 1$.

The following theorem gives a complete characterization of odpu-graphs with radius one.

Theorem 2.13 *A graph with radius 1 and diameter 2 is an odpu-graph if and only if there exists a subset $M \subset V(G)$ with $|M| \geq 2$ such that the induced subgraph $\langle M\rangle$ is complete, $\langle V - M\rangle$ is not complete and any vertex in $V - M$ is adjacent to all the vertices of M.*

Proof Assume that G is an odpu-graph with radius $r = 1$ and diameter $d = 2$. Then, $f^o_M(v) = \{1\}$ for all $v \in V(G)$. If $\langle M\rangle$ is not complete, then there exist two vertices $u, v \in M$ such that $d(u, v) \geq 2$. Hence, both $f^o_M(u)$ and $f^o_M(v)$ contains a number greater than 1, which is not possible. Therefore, $\langle M\rangle$ is complete. Next, if $x \in V - M$ then, since $f^o_M(x) = \{1\}$, x is adjacent to all the vertices of $\langle M\rangle$. Now, if $\langle V - M\rangle$ is complete, then since $\langle M\rangle$ is complete the above argument implies that G is complete, whence diameter of G would be one, a contradiction. Thus, $\langle V - M\rangle$ is not complete.

Conversely assume $\langle M\rangle$ is complete with $|M| \geq 2$, $\langle V - M\rangle$ is not complete and every vertex of $\langle V - M\rangle$ is adjacent to all the vertices in $\langle M\rangle$. Then, clearly, the diameter of G is two and radius of G is one. Also, since $|M| \geq 2$, there exist at least two universal vertices in M (i.e. Each is adjacent to every other vertices in M). Therefore $f^o_M(v) = \{1\}$ for every $v \in V(G)$. Hence G must be an odpu-graph with M as an odpu-set. □

Theorem 2.14 *Let G be a graph of order $n \geq 3$. Then the following are equivalent.*

(*i*) *Every k-element subset of $V(G)$ forms an odpu-set, where $2 \leq k \leq n$.*

(ii) Every 2-element subset of $V(G)$ forms an odpu-set.

(iii) G is complete.

Proof Trivially (i) implies (ii)

If every 2-element subset M of $V(G)$ forms an odpu-set, then $f^o_M(v) = \{1\}$ for all $v \in V(G)$ and hence G is complete.

Obviously (iii) implies (i). □

Theorem 2.15 *Any graph G(may or may not be connected) with $\delta(G) \geq 1$ and having no vertex of full-degree can be embedded into an odpu-graph H with G as an induced subgraph of H of order $|V(G)| + 2$ such that $V(G)$ is an odpu-set of the graph H.*

Proof Let G be a graph with $\delta(G) \geq 1$ and having no vertex of full-degree. Let $u, v \in V(G)$ be any two adjacent vertices and let $a, b \notin V(G)$. Let H be the graph obtained by joining a to b and also, joining a to all vertices of G except u and joining the vertex b to all vertices of G except v. Let $M = V(G) \subset V(H)$. Since a is adjacent to all the vertices except u and $d(a, u) = 2$, implies $f^o_M(a) = \{1, 2\}$. Similarly $f^o_M(b) = \{1, 2\}$. Since u is adjacent to v, $1 \in f^o_M(u)$. Since u does not have full degree, there exists a vertex x, which is not adjacent to u. But (u, b, x) is a path in H and hence $d(u, x) = 2$ in H for all such $x \in V(G)$. Therefore $f^o_M(u) = \{1, 2\}$. Similarly $f^o_M(v) = \{1, 2\}$. Now let $w \in V(G)$, $w \neq u, v$. Now since no vertex w is an isolated vertex and w does not have full-degree, there exist vertices x and y in $V(G)$ such that $wx \in E(H)$ and $wy \notin E(H)$. But then, there exists a path (w, a, y) or (w, b, y) with length 2 in H. Also every vertex which is not adjacent to w is at a distance 2 in H. Therefore $f^o_M(w) = \{1, 2\}$. Hence $f^o_M(x) = \{1, 2\}$ for all $x \in V(H)$. Hence H is an odpu-graph and $V(G)$ is an odpu-set of H. □

Remark 2.16 Bollobás [1] proved that almost all graphs have diameter 2 and almost no graph has a node of full degree. Hence almost no graph has radius one. Since $r(G) \leq d(G)$, almost all graphs have $r(G) = d(G) = 2$, that is, almost all graphs are self-centered with diameter 2. Since self-centered graphs are odpu-graphs, the following corollary is immediate.

Corollary 2.17 *Almost all graphs are odpu-graphs.*

§3. Odpu-Number of a Graph

As we have observed in section 2, if G has an odpu-set M then $M \subseteq Z(G)$ and if $M \subseteq M' \subseteq Z(G)$, then M' is also an odpu-set. This motivates the definition of odpu-number of an odpu-graph.

Definition 3.1 *The Odpu-number of a graph G, denoted by $od(G)$, is the minimum cardinality of an odpu-set in G.*

In this section we characterize odpu-graphs which have odpu-number 2 and also prove that there is no graph with odpu-number 3 and for any positive integer $k \neq 1, 3$, there exists a graph with odpu-number k. We also present several embedding theorems. Clearly,

$$2 \leqslant od(G) \leqslant |Z(G)| \text{ for any odpu} - \text{graph } G. \tag{3.1}$$

Since the upper bound for $|Z(G)|$ is $|V(G)|$, the above inequality becomes,

$$2 \leqslant od(G) \leqslant |V(G)|. \tag{3.2}$$

The next theorem gives a characterization of graphs attaining the lower bound in the above inequality.

Theorem 3.2 *For any graph G, $od(G) = 2$ if and only if there exist at least two vertices $x, y \in V(G)$ such that $d(x) = d(y) = |V(G)| - 1$.*

Proof Suppose that the graph G has an odpu-set M with $|M| = 2$. Let $M = \{x, y\}$. We claim that $d(x) = d(y) = n - 1$, where $n = |V(G)|$. If not, there are two possibilities.

Case 1. $d(x) = n - 1$ and $d(y) < n - 1$.

Since $d(x) = n - 1$, x is adjacent to y. Therefore, $f^o_M(x) = \{1\}$. Also, since $d(y) < n - 1$, it follows that $2 \in f^o_M(w)$ for any vertex w not adjacent to v, which is a contradiction.

Case 2. $d(x) < n - 1$ and $d(y) < n - 1$.

If $xy \in E(G)$, then $f^o_M(x) = f^o_M(y) = \{1\}$ and for any vertex w not adjacent to u, $f^o_M(w) \neq \{1\}$.

If $xy \notin E(G)$, then $1 \notin f^o_M(x)$ and for any vertex w which is adjacent to x, $1 \in f^o_M(w)$, which is a contradiction. Hence $d(x) = d(y) = n - 1$.

Conversely, let G be a graph with $u, v \in V(G)$ such that $d(u) = d(v) = n-1$. Let $M = \{u, v\}$. Then $f^o_M(x) = \{1\}$ for all $x \in V(G)$ and hence M is an odpu-set with $|M| = 2$. □

Corollary 3.3 *For any odpu-graph G if $|M| = 2$, then $\langle M \rangle$ is isomorphic to K_2.*

Corollary 3.4 *$od(K_n) = 2$ for all $n \geqslant 2$.*

Corollary 3.5 *If a (p, q)-graph has an odpu-set M with odpu-number 2, then $2p - 3 \leq q \leq \frac{p(p-1)}{2}$.*

Proof By Theorem 3.2, there exist at least two vertices having degree $p - 1$ and hence $q \geq 2p - 3$. The other inequality is trivial. □

Theorem 3.6 *There is no graph with odpu-number three.*

Proof Suppose there exists a graph G with $od(G) = 3$ and let $M = \{x, y, z\}$ be an odpu-set in G. Since G is connected, $1 \in f^o_M(x) \cap f^o_M(y) \cap f^o_M(z)$.

We claim that x, y, z form a triangle in G. Since $1 \in f^o_M(x)$, and $1 \in f^0_M(z)$, we may assume that $xy, yz \in E(G)$. Now if $xz \notin E(G)$, then $d(x, z) = 2$ and hence $2 \in f^o_M(x) \cap f^0_M(Z)$ and $f^o_M(y) = \{1\}$, which is not possible. Thus $xz \in E(G)$ and x, y, z forms a triangle in G.

Now $f_M^o(w) = \{1\}$ for any $w \in V(G) - M$ and hence w is adjacent to all the vertices of M. Thus G is complete and $od(G) = 2$, which is again a contradiction. Hence there is no graph G with $od(G) = 3$. □

Next we prove that the existence of graph with odpu-numbers $k \neq 1, 3$. We need the following definition.

Definition 3.7 *The shadow graph $S(G)$ of a graph G is obtained from G by adding for each vertex v of G a new vertex v', called the shadow vertex of v, and joining v' to all the neighbors of v in G.*

Theorem 3.8 *For every positive integer $k \neq 1, 3$, there exists a graph G with odpu-number k.*

Proof Clearly $od(P_2) = 2$ and $od(C_4) = 4$. Now we will prove that the shadow graph of any complete graph K_n, $n \geq 3$ is an odpu-graph with odpu-number $n + 2$.

Let the vertices of the complete graph K_n be $v_1, v_2, \ldots, v_n$ and the corresponding shadow vertices be $v_1^{'}, v_2^{'}, \cdots, v_n^{'}$. Since the shadow graph $S(K_n)$ of K_n is self-centered with radius 2 and $n \geq 3$, by Corollary 2.3, it is an odpu-graph. Let M be the smallest odpu-set of $S(K_n)$. We establish that $|M| = n + 2$ in the following three steps.

First, we show $\{v_1^{'}, v_2^{'}, \cdots, v_n^{'}\} \subseteq M$. If there is a shadow vertex $v_i^{'} \notin M$, then $2 \notin f_M^o(v_i)$ since v_i is adjacent to all the vertices of $S(K_n)$ other than $v_i^{'}$, implying thereby that M is not an odpu-set, contrary to our assumption. Thus, the claim holds.

Now, we show that $M = \{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$ is not an odpu-set of $S(K_n)$. Note that $v_1^{'}, v_2^{'}, \ldots, v_n^{'}$ are pairwise non-adjacent and if $M = \{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$, then $1 \notin f_M^o(v_i^{'})$ for all $v_i^{'} \in M$. But $1 \in f_M^o(v_i)$, $1 \leq i \leq n$, and hence M is not an odpu-set.

From the above two steps, we conclude that $|M| > n$. Now, $M = \{v_1^{'}, v_2^{'}, \ldots, v_n^{'}\} \cup \{v_i\}$ where v_i is any vertex of K_n is not an odpu-set. Further, since all the shadow vertices are pairwise nonadjacent and v_i is not adjacent to $v_i^{'}$, $1 \notin f_M^o(v_i^{'})$. Hence $|M| > n + 1$. Let v_i, $v_j \in V(K_n)$ be any two vertices of K_n and let $M = \{v_i, v_j, v_1^{'}, v_2^{'}, \ldots, v_n^{'}\}$. We prove that M is an odpu-set and thereby establish that $od(G) = n + 2$. Now, $d(v_i, v_j) = 1$ and $d(v_i, v_i^{'}) = d(v_j, v_j^{'}) = 2$, so that $f_M^o(v_i) = f_M^o(v_j) = \{1, 2\}$. Also, for any vertex $v_k \in V(K_n)$, $d(v_k, v_i) = 1$ and $d(v_k, v_k^{'}) = 2$, so that $f_M^o(v_k) = \{1, 2\}$. Again, $d(v_i^{'}, v_j) = d(v_j^{'}, v_i) = 1$ and for any shadow vertex $v_k^{'} \in V(S(K_n))$, $d(v_k^{'}, v_i) = 1$ and since all the shadow vertices are pairwise non-adjacent, $f_M^o(v_k^{'}) = \{1, 2\}$. Thus, M is an odpu-set and $od(G) = n + 2$. □

Remark 3.9 We have proved that 3 cannot be the odpu number of any graph. Hence, by the above theorem, for an odpu-graph the numbers 1 and 3 are the only two numbers forbidden as odpu-numbers of any graph.

Theorem 3.10 $od(C_{2k+1}) = 2k$.

Proof Let $C_{2k+1} = (v_1, v_2, \ldots, v_{2k+1}, v_1)$. Clearly $M = \{v_1, v_2, \ldots, v_{2k}\}$ is an odpu-set of C_{2k+1}. Now, let M be any odpu-set of C_{2k+1}. Then, there exists a vertex $v_i \in V(C_{2k+1})$ such that $v_i \notin M$. Without loss of generality, assume that $v_i = v_{2k+1}$. Then, since $1 \in f_M^o(v_{2k+1})$, either $v_{2k} \in M$ or $v_1 \in M$ or both $v_1, v_{2k} \in M$. Without loss of generality, let $v_1 \in M$. Since

$d(v_1, v_{2k+1}) = 1$ and $v_{2k+1} \notin M$, and v_2 is the only element other than v_{2k+1} at a distance 1 from v_1, we see that $v_2 \in M$. Now, $d(v_2, v_{2k+1}) = 2$ and $v_{2k+1} \notin M$, and v_4 is the only element other than v_{2k+1} at a distance 2; this implies $v_4 \in M$. Proceeding in this manner, we get $v_2, v_4 \ldots, v_{2k} \in M$. Now since $d(v_{2k}, v_{2k+1}) = 1$ and $v_{2k+1} \notin M$, and v_{2k-1} is the only element other than v_{2k+1} at a distance 1 from v_{2k}, we get $v_{2k-1} \in M$. Next, since $d(v_{2k-1}, v_{2k+1}) = 2$ and $v_{2k+1} \notin M$, and v_{2k-3} is the only element other than v_{2k+1} at a distance 2 from v_{2k-1}, we get $v_{2k-3} \in M$. Proceeding like this, we get $M = \{v_1, v_2, \ldots, v_{2k}\}$. Hence $od(C_{2k+1}) = 2k$. □

Definition 3.11([2]) *A graph is an r-decreasing graph if $r(G - v) = r(G) - 1$ for all $v \in V(G)$.*

We now proceed to characterize odpu-graphs G with $od(G) = |V(G)|$. We need the following lemma.

Lemma 3.12 *Let G be a self-centered graph with $r(G) \geq 2$. Then for each $u \in V(G)$, there exist at least two vertices in every i^{th} neighborhood $N_i(u) = \{v \in V(G) : d(u, v) = i\}$ of u, $i = 1, 2, \ldots, r - 1$.*

Proof Let G be a self-centered graph and let u be any arbitrary vertex of G. If possible, let for some i, $1 \leq i \leq r - 1$, $N_i(u)$ contains exactly one vertex, say w. Then, since $e(w) = r$, there exists $x \in V(G)$ such that $d(x, w) = r$.

If $x \in N_j(u)$ for some $j > i$, then $d(u, x) > r$, which is a contradiction. Again if $x \in N_j(u)$ for some $j < i$, then $d(x, w) = r < i \leq r - 1$, which is again a contradiction. Hence $N_i(u)$ contains at least two vertices. □

Theorem 3.13 *Let G be a graph of order n, $n \geq 4$. Then the following conditions are equivalent.*

(i) $od(G) = n$.

(ii) the graph G is self-centered with radius $r \geq 2$ and for every $u \in V(G)$, there exists exactly one vertex v such that $d(u, v) = r$.

(iii) the graph G is r-decreasing.

(iv) there exists a decomposition of $V(G)$ into pairs $\{u, v\}$ such that $d(u, v) = r(G) >$ max $(d(u, x), d(x, v))$ for every $x \in V(G) - \{u, v\}$.

Proof Let G be a graph of order n, $n \geq 4$. The equivalence of $(ii), (iii)$ and (iv) follows from Theorem 1.1. We now prove that (i) and (ii) are equivalent.

$(i) \Rightarrow (ii)$

Let G be a graph with $od(G) = n = |V(G)|$. Hence, $e(u) = r$ for all $u \in V(G)$ so that G is self-centered. Now, we show that for every $u \in V(G)$, there exists exactly one vertex $v \in V(G)$ such that $d(u, v) = r$.

First, we show that for some vertex $u_0 \in V(G)$, there exists exactly one vertex $v_0 \in V(G)$ such that $d(u_o, v_0) = r$. Suppose for every vertex $x \in V(G)$, there exist at least two vertices x_1 and x_2 in $V(G)$ such that $d(x, x_1) = r$ and $d(x, x_2) = r$. Let $M = V(G) - \{x_1\}$. Then, since $d(x, x_2) = r$, $f^o_M(x) = \{1, 2, \ldots, r\}$. Further, since $d(x, x_1) = r$, $f^o_M(x_1) = \{1, 2, \ldots, r\}$. Also, since $d(x, x_2) = r$, and by Lemma 3.12, $f^o_M(x_2) = \{1, 2, \cdots, r\}$. Let y be any vertex other than

x, x_1 and x_2. Let $1 \leqslant k \leqslant r$, and if $d(y,x) = k$, then by Lemma 3.12 and by assumption, there exists another vertex $z \in M$ such that $d(y,z) = k$. Therefore, $f_M^o(y) = \{1, 2, \ldots, r\}$. Thus $M = V(G) - \{x_1\}$ is an odpu-set for G, which is a contradiction to the hypothesis. Thus, there exists a vertex $u_0 \in V(G)$ such that there is exactly one vertex $v_0 \in V(G)$ with $d(u_0, v_0) = r$. Next, we claim that u_0 is the unique vertex for v_0 such that $d(u_0, v_0) = r$. Suppose there is a vertex $w_0 \neq u_0$ with $d(w_0, v_0) = r$. Let $M = V(G) - \{u_0\}$. Then, $d(u_0, v_0) = r$ implies $f_M^o(u_0) = \{1, 2, \ldots, r\}$ and $d(v_0, w_0) = r$ imply $f_M^o(v_0) = \{1, 2, \ldots, r\}$. Also, since $d(v_0, w_0) = r$, by Lemma 3.12, it follows that $f_M^o(w_0) = \{1, 2, \ldots, r\}$. Now let $x \in V(G) - \{u_0, v_0, w_0\}$. Since $d(x, u_0) < r$, we get $f_M^o(x) = \{1, 2, \ldots, r\}$. Hence, $M = V(G) - \{u_0\}$ is an odpu-set for G, which is a contradiction. Therefore, for the vertex v_0, u_0 is the unique vertex such that $d(u_0, v_0) = r$.

Next, we claim that there is some vertex $u_1 \in V(G) - \{u_0, v_0\}$ such that there is exactly one vertex $v_1 \in V(G)$ at a distance r from u_1. If for every vertex $u_1 \in V(G) - \{u_0, v_0\}$, there are at least two vertices v_1 and w_1 in $V(G)$ at a distance r from u_1, then proceeding as above, we can prove that $M = V(G) - \{v_1\}$ is an odpu-set of G, a contradiction. Therefore, v_1 is the only vertex at a distance r from u_1. Continuing the above procedure we conclude that for every vertex $u \in V(G)$ there exists exactly one vertex $v \in V(G)$ at a distance r from u and for the vertex v, u is the only vertex at a distance r. Thus (i) implies (ii).

Now, suppose (ii) holds. Then M is the unique odpu-set of G and hence $od(G) = n$. □

Corollary 3.14 *If G is an odpu-graph with $od(G) = |V(G)| = n$, then G is self-centered and n is even.*

Corollary 3.15 *If G is an odpu-graph with $od(G) = |V(G)| = n$ then $r(G) \geq 3$ and u_1, u_2 are different vertices of G, then, $N(u_1) \neq N(u_2)$.*

Proof If $N(u_1) = N(u_2)$, then $d(u_1, v_1) = d(u_2, v_1)$, which contradicts Theorem 3.13. □

Corollary 3.16 *The odpu-number $od(G) = |V(G)|$ for the n-dimensional cube and for even cycle C_{2n}.*

Corollary 3.17 *Let G be a graph with $r(G) = 2$. Then $od(G) = |V(G)|$ if and only if G is isomorphic to $K_{2,2,\ldots,2}$.*

Proof If $G = K_{2,2,\ldots,2}$, then $r(G) = 2$ and G is self-centered and by Theorem 3.13, $od(G) = |V(G)| = 2n$.

Conversely, let G be a graph with $r(G) = 2$. Then G is self-centered and it follows from Theorem 3.13 that for each vertex, there exists exactly one vertex at a distance 2. Hence $G \cong K_{2,2,\ldots,2}$. □

Problem 3.1 *Characterize odpu-graphs for which $od(G) = |Z(G)|$.*

Theorem 3.18 *If a graph G has odpu-number 4, then $r(G) = 2$.*

Proof Let G be an odpu-graph with odpu-number 4. Let $M = \{u, v, x, y\}$ be an odpu-set of G. If $r(G) = 1$, then $f_M^o(x) = \{1\}$ for all $x \in V(G)$. Therefore, $\langle M \rangle$ is complete. Hence, any two elements of M forms an odpu-set of G which implies $od(G) = 2$, which is a contradiction.

Hence $r(G) \geq 2$.

Since $r(G) \geq 2$, none of the vertices in M is adjacent to all the other vertices in M and $\langle M \rangle$ has no isolated vertex. Hence $\langle M \rangle = P_4$ or C_4 or $2K_2$.

If $\langle M \rangle = P_4$ or C_4 then the radius of $\langle M \rangle$ is 2. Hence, there exists a vertex v in M such that $f^o_M(v) = \{1, 2\}$ so that $r(G) = 2$.

Suppose $\langle M \rangle = 2P_2$ and let $E(\langle M \rangle) = \{uv, xy\}$. Since $|M| = 4$, $r(G) \leq 3$. If $r(G) = 3$, then $3 \in f^o_M(x)$ and $3 \in f^o_M(u)$. Hence, there exists a vertex $w \notin M$ such that $xw, uw \in E(G)$. Hence, $d(x, w) = d(u, w) = 1$. Also, $d(y, w) = d(v, w) = 2$. Therefore, $3 \notin f^o_M(w)$, which is a contradiction. Thus, $r(G) = 2$. □

A set S of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex of G is either in S or is adjacent to a vertex in S; further, if $\langle S \rangle$ is isolate-free then S is called a *total dominating set* of G (see Haynes *et al*[7]). The next result establishes the relation between odpu-sets and total dominating sets in an odpu-graph.

Theorem 3.19 *For any odpu-graph G, every odpu-set in G is a total dominating set of G.*

Proof Let M be an odpu-set of the graph G. Since $1 \in f^o_M(u)$, for all $u \in V(G)$, for any vertex $u \in V(G)$ there exists a vertex $v \in M$ such that $uv \in E(G)$. Hence, M is a total dominating set of G. □

Recall that the total domination number $\gamma_t(G)$ of a graph G is the least cardinality of a total dominating set in G.

Corollary 3.20 *For any odpu-graph G, $\gamma_t(G) \leqslant od(G)$.*

Problem 3.2 *Characterize odpu-graphs G such that $\gamma_t(G) = od(G)$.*

Let H be a graph with vertex set $\{x_1, x_2, \ldots, x_n\}$ and let $G_1, G_2, \ldots, G_n$ be a set of vertex disjoint graphs. Then the graph obtained from H by replacing each vertex x_i of H by the graph G_i and joining all the vertices of G_i to all the vertices of G_j if and only if $x_i x_j \in E(H)$, is denoted as $H[G_1, G_2, \ldots, G_n]$.

Theorem 3.21 *Let H be a connected odpu-graph of order $n \geq 2$ and radius $r \geq 2$. Let $K = H[G_1, G_2, \ldots, G_n]$. Then $od(H) = od(K)$.*

Proof Let $V(H) = \{x_1, x_2, \ldots, x_n\}$. Let G_i be the graph replaced at the vertex x_i in H. It follows from the definition of K that if $(x_{i1}, x_{i2}, \ldots, x_{ir})$ is a shortest path in H, then $(x_{i1,j1}, x_{i2,j2}, \ldots, x_{ir,jr})$ is a shortest path in K where $x_{ik,jk}$ is an arbitrary vertex in G_{ik}. Hence $M \subseteq V(H)$is odpu-set in H if and only if the set $M_1 \subseteq V(K)$, where M_1 has exactly one vertex from G_i if and only if $x_i \in M$, is an odpu-set for K. Hence $od(H) = od(K)$. □

Corollary 3.22 *A graph G with radius $r(G) \geq 2$ is an odpu-graph if and only if its shadow graph is an odpu-graph.*

Theorem 3.23 *Given a positive integer $n \neq 1, 3$, any graph G can be embedded as an induced subgraph into an odpu-graph K with odpu-number n.*

Proof If $n=2$, then $K=C_3[G,K_1,K_1]$ is an odpu-graph with $od(K)=od(C_3)=2$ and G is an induced subgraph of K. Suppose $n\geq 4$. Then by Theorem 3.8, there exists an odpu-graph H with $od(H)=n$. Now by Theorem 3.21, $K=H[G,K_1,K_1,\cdots,K_1]$ is an odpu-graph with $od(K)=od(H)=n$ and G is an induced subgraph of K. □

Remark 3.24 If G and K are as in Theorem 3.23, we have

(1) $\omega(H)=\omega(G)+2$,

(2) $\chi(H)=\chi(G)+2$,

(3) $\beta_1(H)=\beta_1(G)+1$ and

(4) $\beta_0(H)=\beta_0(G)$

where $\omega(G)$ is the clique number, $\chi(G)$ is the chromatic number, $\beta_1(G)$ is the matching number and $\beta_0(G)$ is the independence number of G. Since finding these parameters are NP-complete for graphs, finding these four parameters for an odpu-graph is also NP-complete.

§4. Bipartite Odpu-Graphs

In this section we characterize complete multipartite odpu-graphs and bipartite odpu-graphs with odpu-number 2 and 4. Further we prove that there are no bipartite graph with odpu-number 5.

Theorem 4.1 *The complete n-partite graph $K_{a_1,a_2,\cdots,a_n}$ is an odpu-graph if and only if either $a_i=a_j=1$ for some i and j or $a_1,a_2,a_3,\cdots a_n\geq 2$. Hence $od(K_{a_1,a_2,\cdots,a_n})=2$ or $2n$.*

Proof Suppose $G=K_{a_1,a_2,\cdots,a_n}$ is an odpu-graph. If $a_1=1$ for exactly one i, then $|Z(K_{a_1,a_2,\cdots,a_n})|=1$. Hence G is not an odpu-graph, which is a contradiction.

Conversely assume, either $a_i=a_j=1$ for some i and j or $a_1,a_2,a_3,\cdots a_n\geq 2$. If $a_i=a_j=1$ for some i and j, then there exist two vertices of full degree and hence G is an odpu-graph with odpu-number 2. If $a_1,a_2,a_3,\cdots a_n\geq 2$, then for any set M which contains exactly two vertices from each partite set, we have $f^o_M(v)=\{1,2\}$ for all $v\in V(G)$ ane hence M is an odpu-set with $|M|=2n$. Further if M is any subset of $V(G)$ with $|M|<2n$, there exists a partite set V_i such that $|M\cap V_i|\leq 1$ and $f^0_M(v)=\{1\}$ for some $v\in V_i$ and M is not an odpu-set. Hence $od(G)=2n$. □

Theorem 4.2 *Let G be a bipartite odpu-graph. Then $od(G)=2$ if and only if G is isomorphic to P_2.*

Proof Let G be a bipartite odpu-graph with bipartition (X,Y). Let $od(G)=2$. Then, by Theorem 3.2, there exist at least two vertices of degree $n-1$. Hence $|X|=|Y|=1$ and G is isomorphic to P_2. The converse is obvious. □

Theorem 4.3 *A bipartite odpu-graph G with bipartition (X,Y) has odpu-number 4 if and only if the set X has at least two vertices of degree $|Y|$ and the set Y has at least two vertices of degree $|X|$.*

Proof Suppose $od(G)=4$. Let M be an odpu-set of G with $|M|=4$. Then, by Theorem 3.18, $r(G)=2$ and hence $f^o_M(x)=\{1,2\}$ for all $x\in V(G)$.

First, we show that $|M\cap X|=|M\cap Y|=2$. If $|M\cap X|=4$, then $1\notin f^o_M(v)$ for all $v\in M$. If $|M\cap X|=3$ and $|M\cap Y|=1$ then $2\notin f^o_M(v)$ for the vertex $v\in M\cap Y$. Hence it follows that $|M\cap X|=|M\cap Y|=2$. Let $M\cap X=\{u,v\}$ and $M\cap Y=\{x,y\}$. Since $f^0_M(w)=\{1,2\}$ for all $w\in V$, it follows that every vertex in X is adjacent to both x and y and every vertex in Y is adjacent to both u and v. Hence, $deg(u)=deg(v)=|Y|$ and $deg(x)=deg(y)=|X|$.

Conversely, suppose $u,v\in X$, $x,y\in Y$, $deg(u)=deg(v)=|Y|$ and $deg(x)=deg(y)=|X|$. Let $M=\{u,v,x,y,\}$. Clearly $f^0_M(w)=\{1,2\}$ for all $w\in V$. Hence M is an odpu-set. Also, since there exists no full degree vertex in G, by Theorem 3.2 the odpu-number cannot be equal to 2. Also, since 3 is not the odpu-number of any graph. Hence the odpu-number of G is 4. □

Theorem 4.4 *The number 5 cannot be the odpu-number of a bipartite graph.*

Proof Suppose there exists a bipartite graph G with bipartition (X,Y) and $od(G)=5$. Let $M=\{u,v,x,y,z\}$ be a odpu-set for G.

First, we shall show that $|X\cap M|\geq 2$ and $|Y\cap M|\geq 2$. Suppose, on the contrary, one of these inequalities fails to hold, say $|X\cap M|\leq 1$. If X has no element in M, then $1\notin f^o_M(a)$ for all $\in M$, which is a contradiction. Therefore, $|X\cap M|=1$. Without loss of generality, let $\{u\}=X\cap M$. Then, since $1\in f^o_M(v)\cap f^o_M(x)\cap f^o_M(y)\cap f^o_M(z)$, all the vertices v,x,y,z should be adjacent to u. Hence $2\notin f^o_M(u)$, a contradiction. Thus, we see that each of X and Y must have at least two vertices in M. Without loss of generality, we may assume $u,v\in X$ and $x,y,z\in Y$.

Case 1. $r(G)=2$.

Then $f^o_M(w)=\{1,2\}$ for all $w\in Y$. Then proceeding as in Theorem 4.3, we get $deg(u)=deg(v)=|Y|$ and $deg(x)=deg(y)=deg(z)=|X|$. Therefore, by Theorem 4.3, $\{u,v,x,y\}$ forms an odpu-set of G, a contradiction to our assumption that M is a minimum odpu-set of G. Therefore, $r=2$ is not possible.

Case 2. $r(G)\geq 3$.

Since M is an odpu-set of G, $f^o_M(a)=\{1,2,\ldots,r\}$ for all $a\in V(G)$. Then, since $2\in f^o_M(u)$, there exists a vertex $b\in Y$ such that $ub,bv\in E(G)$. But since $b\in Y$ and $ub,bv\in E(G)$, $3\notin f^o_M(b)$, which is a contradiction. Hence the result follows. □

Conjecture 4.5 *For a bipartite odpu-graph the odpu-number is always even.*

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