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# On Pathos Total Semitotal and Entire Total Block Graph of a Tree

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**Abstract**: In this communication, the concept of pathos total semitotal and entire total block graph of a tree is introduced. Its study is concentrated only on trees. We present a characterization of graphs whose pathos total semitotal block graphs are planar, maximal outerplanar, minimally nonouterplanar, nonminimally nonouterplanar, noneulerian and hamiltonian. Also, we present a characterization of those graphs whose pathos entire total block graphs are planar, maximal outerplanr, minimally nonouterplanar, nonminimally nonouterplanar, noneulerian, hamiltonian and graphs with crossing number one.

**Key Words**: Pathos, path number, Smarandachely block graph, semitotal block graph, total block graph, pathos total semitotal block graph, pathos entire total block graph, pathos length, pathos point, inner point number.

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## §1. Introduction

The concept of pathos of a graph $G$ was introduced by Harary [2], as a collection of minimum number of line disjoint open paths whose union is $G$. The path number of a graph $G$ is the number of paths in a pathos. A new concept of a graph valued functions called the semitotal and total block graph of a graph was introduced by Kulli [5]. For a graph $G(p,q)$ if $B = u_1, u_2, u_3, \cdots, u_r; r \geqslant 2$ is a block of $G$. Then we say that point $u_1$ and block $B$ are incident with each other, as are $u_2$ and $B$ and soon. If two distinct blocks $B_1$ and $B_2$ are incident with a common cut point, then they are called adjacent blocks. The points and blocks of a graph are called its members. A *Smarandachely block graph* $T_S^V(G)$ for a subset $V \subset V(G)$ is such a graph with vertices $V \cup \mathcal{B}$ in which two points are adjacent if and only if the corresponding members of $G$ are adjacent in $\langle V \rangle_G$ or incident in $G$, where $\mathcal{B}$ is the set of blocks of $G$. The semitotal block graph of a graph $G$ denoted $T_b(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of $G$ in which two points are adjacent if and only if

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members of $G$ are incident. The total block graph of a graph $G$ denoted by $T_B(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of $G$ in which two points are adjacent if and only if the corresponding members of $G$ are adjacent or incident. Also, a new concept called pathos semitotal and total block graph of a tree has been introduced by Muddebihal [10]. The pathos semitotal graph of a tree $T$ denoted by $P_{T_b}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of $T$ in which two points are adjacent if and only if the corresponding members of $G$ are incident and the lines lie on the corresponding path $P_i$ of pathos. The pathos total block graph of a tree $T$ denoted by $P_{T_B}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of $T$ in which two points are adjacent if and only if the corresponding members of $G$ are adjacent or incident and the lines lie on the corresponding path Pi of pathos. Stanton [11] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected and without loops or multiple lines. The pathos total semitotal block graph of a tree $T$ denoted by is defined as the graph whose point set is the union of set of points and set of blocks of $T$ and the path of pathos of $T$ in which two points are adjacent if and only if the corresponding members of $T$ are incident and the lines lie on the corresponding path $P_i$ of pathos. The pathos entire total block graph of a tree denoted by is defined as the graph whose set of points is the union of set of points, set of blocks and the path of pathos of $T$ in which two points are adjacent if and only if the corresponding members of $T$ are adjacent or incident and the lines lie on the corresponding path $P_i$ of pathos. Since the system of pathos for $T$ is not unique, the corresponding pathos total semitotal block graph and pathos entire total block graphs are also not unique.

In Figure 1, a tree $T$ and its semi total block graph $T_b(T)$ and their pathos total semitotal block graph are shown. In Figure 2, a tree $T$ and its total block graph $T_B(T)$ and their pathos entire total block graphs are shown.

The line degree of a line $uv$ in $T$, pathos length in $T$, pathos point in $T$ was defined by Muddebihal [9]. If $G$ is planar, the inner point number $i(G)$ of $G$ is the minimum number of points not belonging to the boundary of the exterior region in any embedding of $G$ in the plane. A graph $G$ is said to be minimally nonouterplanar if $i(G) = 1$ as was given by Kulli [4].

We need the following results for our further results.

**Theorem** $A$([10]) *For any non-trivial tree $T$, the pathos semitotal block graph $P_{T_b}(T)$ of a tree $T$, whose points have degree $d_i$, then the number of points are $(2q + k + 1)$ and the number of lines are $\left(2q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$, where $k$ is the path number.*

**Theorem** $B$([10]) *For any non-trivial tree whose points have degree $d_i$, the number of points and lines in total block graph $T_B(T)$ of a tree $T$ are $(2q + 1)$ and $\left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$.*

**Theorem** $C$([10]) *For any non-trivial tree $T$, the pathos total block graph $P_{T_B}(T)$ of a tree*

*$T$, whose points have degree $d_i$, then the number of points in $P_{T_B}(T)$ are $(2q+k+1)$ and the number of lines are $\left(q+2+\sum_{i=1}^{p} d_i^2\right)$, where $k$ is the path number.*

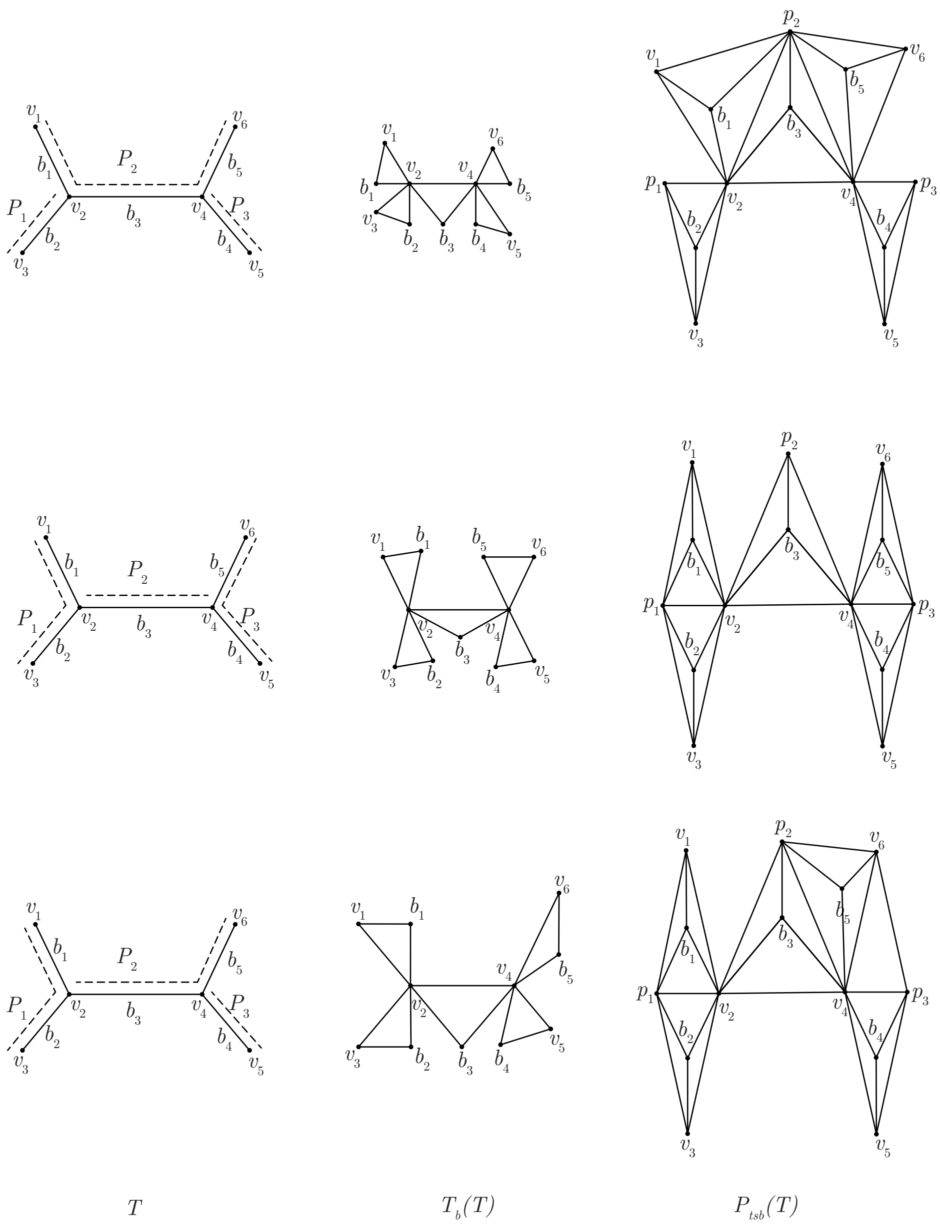


**Figure** 1

**Theorem** $D$([7]) *The total block graph $T_B(G)$ of a graph $G$ is planar if and only if $G$ is outerplanar and every cut point of $G$ lies on at most three blocks.*

**Theorem** $E$([6]) *The total block graph $T_B(G)$ of a connected graph $G$ is minimally nonouter-*

*planar if and only if,*

(1) *G if a cycle, or*

(2) *G is a path of length $n \geqslant 2$, together with a point which is adjacent to any two adjacent points of $P$.*

**Theorem** $F$([8]) *The total block graph $T_B(G)$ of a graph $G$ has crossing number one if and only if,*

(1) *G is outerplanar and every cut point in G lies on at most 4 blocks and G has a unique cut point which lies on 4 blocks, or*

(2)*G is minimally nonouterplanar, every cut point of G lies on at most 3 blocks and exactly one block of G is theta-minimally nonouterplanar.*

**Corollary** $A$([1]) *Every non-trivial tree $T$ contains at least two end points.*

## §2. Pathos Total Semitotal Block Graph of a Tree

We start with a few preliminary results.

**Remark** 2.1 The number of blocks in pathos total semitotal block graph $P_{tsb}(T)$ of a tree $T$ is equal to the number of pathos in $T$.

**Remark** 2.2 If the pathos length of the path $P_i$ of pathos in $T$ is $n$, then the degree of the corresponding pathos point in $P_{etb}(T)$ is $2n + 1$.

In the following theorem we obtain the number of points and lines in pathos total semitotal block graph $P_{tsb}(T)$ of a tree $T$.

**Theorem** 2.1 *For any non-trivial tree $T$, the pathos total semi total block graph $P_{tsb}(T)$ of a tree $T$, whose points have degree $d_i$, then the number of points in $P_{tsb}(T)$ are $(2q + k + 1)$ and the number of lines are*

$$\left(3q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$$

*where $k$ is the path number.*

*Proof* By Theorem A, the number of points in $P_{T_b}(T)$ are $(2q + k + 1)$, and by definition of $P_{tsb}(T)$, the number of points in $P_{tsb}(T)$ are $(2q + k + 1)$, where $k$ is the path number. Also by Theorem A, the number of lines in $P_{T_b}(T)$ are $\left(2q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$. The number of lines in $P_{tsb}(T)$ is equal to the sum of lines in $P_{T_b}(T)$ and the number of lines which lie on the lines (or blocks) of pathos, which are equal to $q$, since the number of lines are equal to the number of blocks in a tree $T$. Thus the number of lines in $P_{tsb}(T)$ is equal to

$$\left[q + (2q + 2 + \frac{1}{2}\sum_{i=1}^{p} {d_i}^2)\right] = 3q + 2 + \frac{1}{2}\sum_{i=1}^{p} {d_i}^2. \qquad \square$$

## §3. Planar Pathos Total Semitotal Block Graphs

A criterion for pathos total semitotal block graph $P_{tsb}(T)$ of a tree $T$ to be planar is presented in our next theorem.

**Theorem** 3.1 *For any non-trivial tree $T$, the pathos total semi total block graph $P_{tsb}(T)$ of a tree $T$ is planar.*

*Proof* Let $T$ be a non-trivial tree, then in $T_b(T)$ each block is a triangle. We have the following cases.

**Case** 1 Suppose $G$ is a path, $G = Pn : u_1, u_2, u_3, \cdots, u_n, n > 1$. Further, $V[T_b(T)] = \{u_1, u_2, u_3, \cdots, u_n, b_1, b_2, b_3, \cdots, b_{n-1}\}$, where $b_1, b_2, b_3, \cdots, b_{n-1}$ are the corresponding block points. In pathos total semi total block graph $P_{tsb}(T)$ of a tree $T$, the pathos point $w$ is adjacent to, $\{u_1, u_2, u_3, \ldots, u_n, b_1, b_2, b_3, \cdots, b_{n-1}\}$. For the pathos total semitotal block graph $P_{tsb}(T)$ of a tree $T, \{u_1b_1u_2w, u_2b_2u_3w, u_3b_3u_4w, \cdots, u_{n-1}b_{n-1}u_nw\} \in V[P_{tsb}(T)]$, in which each set $\{u_{n-1}b_{n-1}u_nw\}$ forms an induced subgraph as $K_4$. Hence one can easily verify that each induced subgraphs of corresponding set $\{u_{n-1}b_{n-1}u_nw\}$ is planar. Hence $P_{tsb}(T)$ is planar.

**Case** 2 Suppose $G$ is not a path. Then $V[T_b(G)] = \{u_1, u_2, u_3, \cdots, u_n, b_1, b_2, b_3, \cdots, b_{n-1}\}$ and $w_1, w_2, w_3, \cdots, w_k$ be the pathos points.Since $u_{n-1}u_n$ is a line and $u_{n-1}u_n = b_{n-1} \in V[T_b(G)]$. Then in $P_{tsb}(G)$ the set $\{u_{n-1}, b_{n-1}, u_n, w\}$ $\forall\, n > 1$, forms $K_4$ as an induced subgraphs. Hence $P_{tsb}(T)$ is planar. □

The next theorem gives a minimally nonouterplanar $P_{tsb}(T)$.

**Theorem** 3.2 *For any non-trivial tree $T$, the pathos total semitotal block graph $P_{tsb}(T)$ of a tree $T$ is minimally nonouterplanar if and only if $T = K_2$.*

*Proof* Suppose $T = K_3$, and $P_{tsb}(T)$ is minimally nonouterplanar, then $T_b(T) = K_4$ and one can easily verify that $i\,(P_{tsb}(T)) > 1$, a contradiction.

Suppose $T \neq K2$. Now assume $T = K_{1,2}$ and $P_{tsb}(T)$ is minimally nonouterplanar. Then $T_b(T) = k_3 \cdot k_3$. Since $K_{1,2}$ has exactly one pathos and let $v$ be a pathos point which is adjacent to all the points of $k_3 \cdot k_3$ in $P_{tsb}(T)$. Then one can easily see that, $i\,(P_{tsb}(T)) > 1$ a contradiction.

For converse, suppose $T = K_2$, then $T_b(T) = K_3$ and $P_{tsb}(T) = K_4$. Hence $P_{tsb}(T)$ is minimally nonouterplanar. □

From Theorem 3.2, we developed the inner point number of a tree as shown in the following corollary.

**Corollary** 3.1 *For any non-trivial tree $T$ with $q$ lines, $i\,(P_{tsb}(T)) = q$.*

*Proof* The result is obvious for a tree with $q = 1$ and 2. Next we show that the result is true for $q \geq 3$. Now we consider the following cases.

**Case 1** Suppose $T$ is a path, $P : v_1, v_2, \ldots, v_n$ such that $v_1v_2 = e_1, v_2v_3 = e_2 \cdots, v_{n-1}v_n =$

$e_{n-1}$ be the lines of $P$. Since each $e_i, 1 \leq i \leq n-1$, be a block of $P$, then in $T_b(P)$, each $e_i$ is a point such that $V[T_b(P)] = V(P) \cup E(P)$. In $T_b(P)$ each $v_1e_1v_2, v_3e_2v_3, \cdots, v_{n-1}e_{n-1}v_n$ forms a block in which each block is $k_3$. Since each line is a block in $P$,then the number of $k_3$'s in $Tb(P)$ is equal to the numbers of lines in $P$. In $P_{tsb}(P)$, it has exactly one pathos. Then $V[P_{tsb}(P)] = V[T_b(P)] \cup \{P\}$ and $P$ together with each block of $T_b(P)$ forms a block as $P_{tsb}(P)$. Now the points $p, v_1, e_1, v_2$ forms $k_4$ as a subgraph of a block $P_{tsb}(P)$. Similarly each $\{v_2, e_2, v_3, p\}, \{v_3, e_3, v_4, p\}, \cdots, \{v_{n-1}, e_{n-1}, v_n, p\}$ forms $k_4$ as a subgraph of a block $P_{tsb}(P)$. One can easily find that each point $e_i, 1 \leq i \leq n-1$ lie in the interior region of $k_4$, which implies that $i\left(P_{tsb}(P)\right) = q$.

**Case** 2 Suppose $T$ is not a path, then $T$ has at least one point of degree greater than two. Now assume $T$ has exactly one point $v$, deg$v \geq 3$. Then $T = K_{1,n}$. If $P_{tsb}(T)$ has inner point number two which is equal to $n = q$. Similarly if $n$ is odd then $P_{tsb}(T)$ has $n-1$ blocks with inner point number two and exactly one block which is isomorphic to $k_4$. Hence $i\left[P_{tsb}\left(K_{1,n}\right)\right] = q$. Further this argument can be extended to a tree with at least two or more points of degree greater two. In each case we have $i\left[P_{tsb}\left(T\right)\right] = q$. □

In the next theorem, we characterize the noneulerian $P_{tsb}(T)$.

**Theorem** 3.3 *For any non-trivial tree $T$, the pathos total semitotal block graph $P_{tsb}(T)$ of a tree $T$ is noneulerian.*

*Proof* We have the following cases.

**Case** 1 Suppose $\Delta(T) \leq 2$ and if $p = 2$ points, then $P_{tsb}(T) = K_4$, which is noneulerian. If $T$ is a path with $p > 2$ points. Then in $T_b(T)$ each block is a triangle such that they are in sequence with the vertices of $Tb(T)$ as $\{v_1, b_1, v_2, v_1\}$ an induced subgraph as a triangle in $T_b(T)$. Further $\{v_2, b_2, v_3, v_2\}, \{v_3, b_3, v_4, v_3\}, \cdots, \{v_{n-1}, b_n, v_n, v_{n-1}\}$, in which each set form a triangle as an induced subgraph of $T_b(T)$. Clearly one can easily verify that $T_b(T)$ is eulerian. Now this path has exactly one pathos point say $k_1$, which is adjacent to $v_1, v_2, v_3, \cdots, v_n, b_1, b_2, b_3, \cdots, b_{n-1}$ in $P_{tsb}(T)$ in which all the points $v_1, v_2, v_3, \ldots, v_n, b_1, b_2, b_3, \cdots, b_{n-1} \in P_{tsb}(T)$ are of odd degree. Hence $P_{tsb}(T)$ is noneulerian.

**Case** 2 Suppose $\Delta(T) \geq 3$. Assume $T$ has a unique point of degree $\geq 3$ and also assume that $T = K_{1,n}$. Then in $T_b(T)$ each block is a triangle, such that there are n number of blocks which are $K_3$ with a common cut point k. Since the degree of a vertex $k = 2n$. One can easily verify that $T_b\left(K_{1,3}\right)$ is eulerian. To form $P_{tsb}(T), T = K_{1,n}$, the points of degree 2 and the point $k$ are joined by the corresponding pathos point which gives points of odd degree in $P_{tsb}(T)$. Hence $P_{tsb}(T)$ is noneulerian. □

In the next theorem we characterize the hamiltonian $P_{tsb}(T)$.

**Theorem** 3.4 *For any non-trivial tree $T$, the pathos semitotal block graph $P_{tsb}(T)$ of a tree $T$ is hamiltonian if and only if $T$ is a path.*

*Proof* For the necessity, suppose $T$ is a path and has exactly one path of pathos.

Let $V\left[T_b\left(T\right)\right] = \{u_1, u_2, u_3, \cdots, u_n\}\{b_1, b_2, b_3, \cdots, b_{n-1}\}$, where $b_1, b_2, b_3, \ldots, b_{n-1}$ are block points of $T$. Since each block is a triangle and each block consists of points as $B_1 = \{u_1, b_1, u_2\}, B_2 = \{u_2, b_2, u_3\}, \cdots, B_m = \{u_m, b_m, u_{m+1}\}$. In $P_{tsb}(T)$ the pathos point w is adjacent to $\{u_1, u_2, u_3, \cdots, u_n, b_1, b_2, b_3, \cdots, b_{n-1}\}$. Hence $V[P_{tsb}(T)] = \{u_1, u_2, u_3, \cdots, u_n\} \cup \{b_1, b_2, b_3, \cdots, b_{n-1}\} \cup w$ form a spanning cycle as w, $u_1, b_1, u_2, b_2, u_2, \cdots, u_{n-1}, b_{n-1}, u_n, w$ of $P_{tsb}(T)$. Clearly $P_{tsb}(T)$ is hamiltonian. Thus the necessity is proved.

For the sufficiency, suppose $P_{tsb}(T)$ is hamiltonian. Now we consider the following cases.

**Case** 1 Assume $T$ is a path. Then $T$ has at least one point with deg$v \geq 3$, $\forall\ v \in V(T)$, suppose $T$ has exactly one point $u$ such that degree $u > 2$ and assume $G = T = K_{1,n}$. Now we consider the following subcases of case 1.

**Subcase** 1.1 For $K_{1,n}, n > 2$ and if $n$ is even, then in $T_b(T)$ each block is $k_3$. The number of path of pathos are $\frac{n}{2}$. Since $n$ is even we get $\frac{n}{2}$ blocks in $P_{tsb}(T)$, each block contains two times of $\langle K_4 \rangle$ with some edges common. Since $P_{tsb}(T)$ has a cut points, one can easily verify that there does not exist any hamiltonian cycle, a contradiction.

**Subcase** 1.2 For $K_{1,n}, n > 2$ and $n$ is odd, then the number of path of pathos are $\frac{n+1}{2}$, since $n$ is odd we get $\frac{n-1}{2} + 1$ blocks in which $\frac{n-1}{2}$ blocks contains two times of $\langle K_4 \rangle$ which is nonline disjoint subgraph of $P_{tsb}(T)$ and remaining blocks is $\langle K_4 \rangle$. Since $P_{tsb}(T)$ contain a cut point, clearly $P_{tsb}(T)$ does not contain a hamiltonian cycle, a contradiction. Hence the sufficient condition. □

## §4. Pathos Entire Total Block Graph of a Tree

A tree T, its total block graph $T_B(T)$, and their pathos entire total block graphs $P_{etb}(T)$ are shown in Figure 2. We start with a few preliminary results.

**Remark** 4.1 If the pathos length of path $P_i$ of pathos in $T$ is $n$, then the degree of the corresponding pathos point in $P_{etb}(T)$ is $2n + 1$.

**Remark** 4.2 For any nontrivial tree $T$, the pathos entire total block graph $P_{etb}(T)$ is a block.

**Theorem** 4.1 *For any non-trivial tree $T$, the pathos total block graph $P_{etb}(T)$ of a tree $T$, whose points have degree $d_i$, then the number of points in $P_{etb}(T)$ are $(2q + k + 1)$ and the number of lines are $\left(2q + 2 + \sum_{i=1}^{p} d_1^2\right)$, where $k$ is the path number.*

*Proof* By Theorem C, the number of points in $P_{T_B}(T)$are $(2q + k + 1)$, by definition of $P_{etb}(T)$, the number of points in $P_{etb}(T)$ are $(2q + k + 1)$, where $k$ is the path number in $T$. Also by Theorem B, the number of lines in $T_B(T)$ are $\left(2q + \frac{1}{2}\sum_{i=1}^{p} {d_i}^2\right)$. By Theorem C, The number of lines in $P_{T_B}(T)$ are $\left(q + 2 + \sum_{i=1}^{p} d_i^2\right)$. By definition of pathos entire total block graph $P_{etb}(T)$ of a tree equal to the sum of lines in $P_{T_B}(T)$ and the number of lines which lie on block points $b_i$ of $T_B(T)$ from the pathos points $P_i$, which are equal to $q$. Thus the number of lines in $P_{etb}(T) = \left(q + 2 + \sum_{i=1}^{p} d_i^2\right) = \left(2q + 2 + \sum_{i=1}^{p} d_1^2\right).$ $\square$

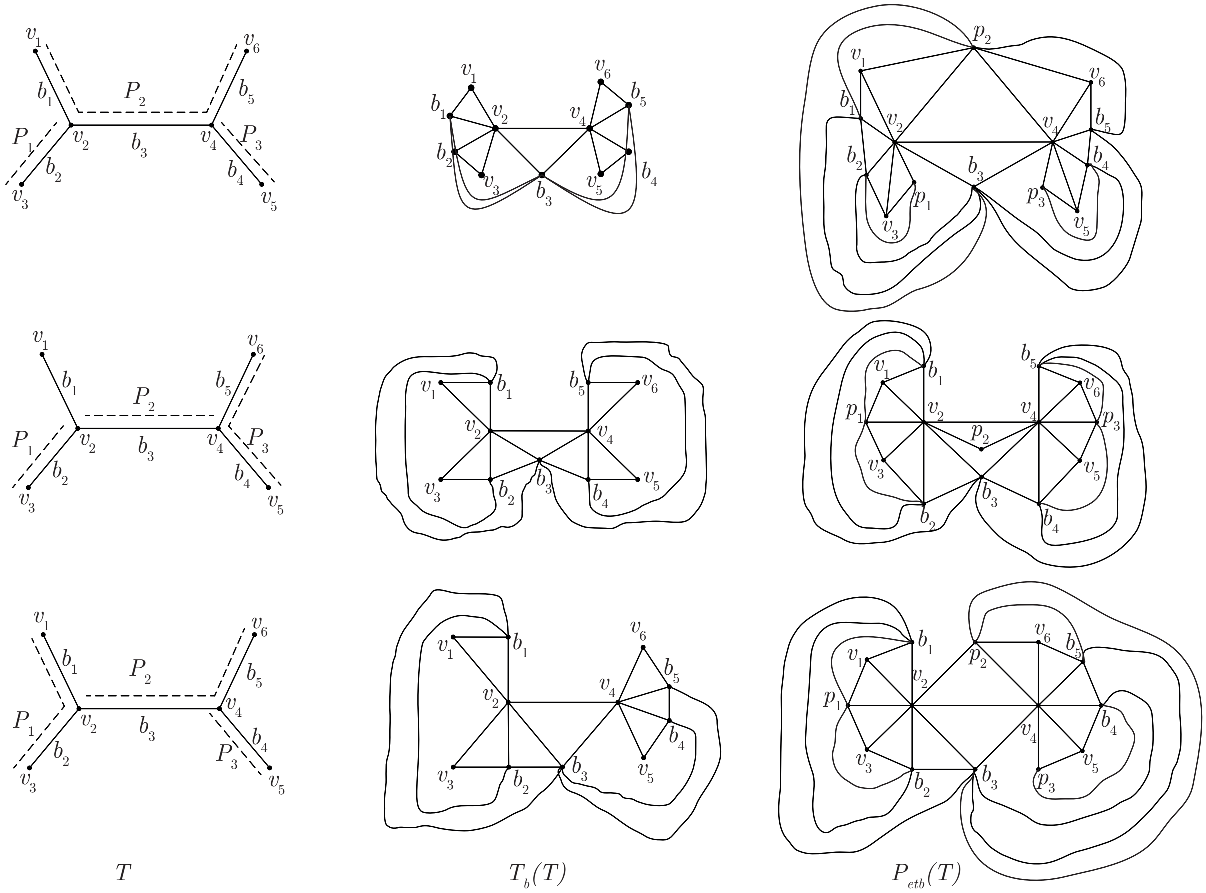


**Figure** 2

## §5. Planar Pathos Entire Total Block Graphs

A criterion for pathos entire total block graph to be planar is presented in our next theorem.

**Theorem** 5.1 *For any non-trivial tree $T$, the pathos entire total block graph $P_{etb}(T)$ of a tree $T$ is planar if and only if $\Delta(T) \leq 3$.*

*Proof* Suppose $P_{etb}(T)$ is planar. Then by Theorem D, each cut point of $T$ lie on at most 3 blocks. Since each block is a line in a tree, now we can consider the degree of cut points instead of number of blocks incident with the cut points. Now suppose if $\Delta(T) \leq 3$, then by Theorem D, $T_B(T)$ is planar. Let $\{b_1, b_2, b_3, \cdots, b_{p-1}\}$ be the blocks of $T$ with $p$ points such that $b_1 = e_1, b_2 = e_2, \cdots, b_{p-1} = e_{p-1}$ and $P_i$ be the number of pathos of $T$. Now $V[P_{etb}(T)] = V(G) \cup b_1, b_2, b_3, \cdots, b_{p-1} \cup \{P_i\}$. By Theorem D, and by the definition of pathos, the embedding of $P_{etb}(T)$ in any plane gives a planar $P_{etb}(T)$.

Conversely, Suppose $\Delta(T) \geq 4$ and assume that $P_{etb}(T)$ is planar. Then there exists at least one point of degree 4, assume that there exists a vertex $v$ such that deg$v = 4$. Then in $T_B(T)$, this point together with the block points form $k_5$ as an induced subgraph. Further the corresponding pathos point which is adjacent to the $V(T)$ in $T_B(T)$ which gives $P_{etb}(T)$ in which again $k_5$ as an induced subgraph, a contradiction to the planarity of $P_{etb}(T)$. This completes the proof. □

The following theorem results the minimally nonouterplanar $P_{etb}(T)$.

**Theorem** 5.2 *For any non-trivial tree $T$, the pathos entire total block graph $P_{etb}(T)$ of a tree $T$ is minimally nonouterplanar if and only if $T = k_2$.*

*Proof* Suppose $T = k_3$ and $P_{etb}(T)$ is minimally nonouterplanar. Then $T_B(T) = k_4$ and one can easily verify that, $i(P_{etb}(T)) > 1$, a contradiction. Further we assume that $T = K_{1,2}$ and $P_{etb}(T)$ is minimally outerplanar, then $T_B(T)$ is $W_p - x$, where x is outer line of $W_p$. Since $K_{1,2}$ has exactly one pathos, this point together with $W_p - x$ gives $W_{p+1}$. Also in $P_{etb}(T)$ and by definition of $P_{etb}(T)$ there are two more lines joining the pathos points there by giving $W_{p+3}$. Clearly, $P_{etb}(T)$ is nonminimally nonouterplanar, a contradiction.

For the converse, if $T = k_2, T_B(T) = k_3$ and $P_{etb}(T) = K_4$ which is a minimally nonouterplanar. This completes the proof of the theorem. □

Now we have a pathos entire total block graph of a path $p \geq 2$ point as shown in the below remark.

**Remark** 5.1 For any non-trivial path with $p \geq 2$ points, $i[P_{etb}(T)] = 2p - 3$. The next theorem gives a nonminimally nonouterplanar $P_{etb}(T)$.

**Theorem** 5.3 *For any non-trivial tree $T$, the pathos entire total block graph $P_{etb}(T)$ of a tree $T$ is nonminimally nonouterplanar except for $T = k_2$.*

*Proof* Assume $T$ is not a path. We consider the following cases.

**Case** 1 Suppose $T$ is a tree with $\Delta(T) \geq 3$. Then there exists at least one point of degree at least 3. Assume $v$ be a point of degree 3. Clearly, $T = K_{1,3}$. Then by the Theorem F, $i\,[T_B(T)] > 1$. Since $T_B(T)$ is a subgraph of $P_{etb}(T)$. Clearly, $i\,(P_{etb}(T)) \geqslant 2$. Hence $P_{etb}(T)$ is nonminimally nonouterplanar.

**Case** 2 Suppose $T$ is a path with $p$ points and for $p > 2$ points. Then by Remark 5.1, $i\,[P_{etb}(T)] > 1$. Hence $P_{etb}(T)$ is nonminimally nonouterplanar. □

In the following theorem we characterize the noneulerian $P_{etb}(T)$.

**Theorem** 5.4 *For any non-trivial tree $T$, the pathos entire total block graph $P_{etb}(T)$ of a tree $T$ is noneulerian.*

*Proof* We consider the following cases.

**Case** 1 Suppose $T$ is a path $P_n$ with $n$ points. Now for $n = 2$ and 3 points as follows. For $p = 2$ points, then $P_{etb}(T) = K_4$, which is noneulerian. For $p = 3$ points, then $P_{etb}(T)$ is a wheel $W_6$ together with two lines joining the non adjacent points in which one point is common for these two lines as shown in the Figure 3, which is noneulerian.

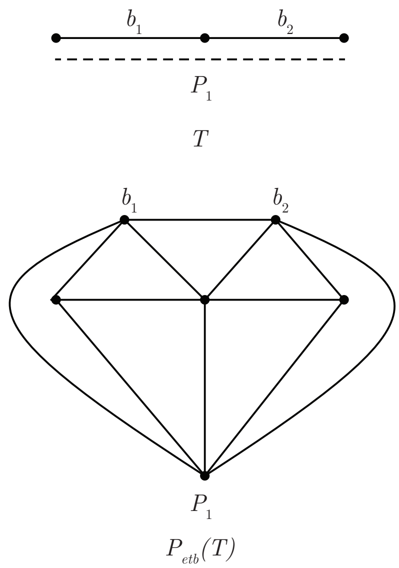


**Figure** 3

For $p \geq 4$ points, we have a path $P : v_1, v_2, v_3, \ldots, v_p$. Now in path each line is a block. Then $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \ldots, v_{p-1}v_p = e_{p-1} = b_{p-1}$, $\forall\ e_{p-1} \in E(G)$, and $\forall\ b_{p-1} \in V[T_B(P)]$. In $T_B\,(P)$, the degree of each point is even except $b_1$ and $b_{p-1}$. Since the path $P$ has exactly one pathos which is a point of $P_{etb}(P)$ and is adjacent to the points $v_1, v_2, v_3, \ldots, v_p$, of $T_B\,(P)$ which are of even degree, gives as an odd degree points in $P_{etb}(P)$ including odd degree points $b_1$ and $b_{p-1}$. Clearly $P_{etb}(P)$ is noneulerian.

**Case** 2 Suppose $T$ is not a path. We consider the following subcases.

**Subcase** 2.1 Assume $T$ has a unique point degree $\geq 3$ and $T = K_{1,n}$, with $n$ is odd. Then in $T_B\,(T)$ each block is a triangle such that there are n number of triangles with a common cut points $k$ which has a degree $2n$. Since the degree of each point in $T_B\,(K_{1,n})$ is odd other than the cut point $k$ which are of degrees either 2 or $n + 1$. Then $P_{etb}(T)$ eulerian. To form $P_{etb}(T)$ where $T = K_{1,n}$, the points of degree 2 and 4 the point $k$ are joined by the corresponding pathos point which gives $(2n + 2)$ points of odd degree in $P_{etb}(K_{1,n})$. $P_{etb}(T)$ has n points of odd degree. Hence $P_{etb}(T)$ noneulerian.

Assume that $T = K_{1,n}$, where $n$ is even, Then in $T_B\,(T)$ each block is a triangle, which are $2n$ in number with a common cut point $k$. Since the degree of each point other than $k$ is either 2 or $(n + 1)$ and the degree of the point $k$ is $2n$. One can easily verify that $T_B\,(K_{1,n})$ is noneulerian. To form $P_{etb}(T)$ where $T = K_{1,n}$, the points of degree 2 and 5 the point $k$ are joined by the corresponding pathos point which gives $(n + 2)$ points of odd degree in $P_{etb}(T)$.

Hence $P_{etb}(T)$ noneulerian.

**Subcase** 2.2 Assume $T$ has at east two points of degree $\geq 3$. Then $V[T_B(T)] = V(G) \cup b_1, b_2, b_3, \ldots, b_p$, $\forall\ b_p \in E(G)$. In $T_B\,(T)$, each endpoint has degree 2 and these points are adjacent to the corresponding pathos points in $P_{etb}(T)$ gives degree 3, From case 1, Tree $T$ has at least 4 points and by Corollary [A], $P_{etb}(T)$ has at least two points of degree 3. Hence $P_{etb}(T)$ is noneulerian. □

In the next theorem we characterize the hamiltonian $P_{etb}(T)$.

**Theorem** 5.5 *For any non-trivial tree $T$, the pathos entire total block graph $P_{etb}(T)$ of a tree $T$ is hamiltonian.*

*Proof* we consider the following cases.

**Case** 1 Suppose $T$ is a path with $\{u_1, u_2, u_3, \ldots, u_n\} \in V(T)$ and $b_1, b_2, b_3, \ldots, b_m$ be the number of blocks of $T$ such that $m = n - 1$. Then it has exactly one path of pathos. Now point set of $T_B\,(T)$, $V\,[T_B(T)] = \{u_1, u_2, \cdots, u_n\} \cup \{b_1, b_2, \ldots, b_m\}$. Since given graph is a path then in $T_B\,(T)$, $b_1 = e_1, b_2 = e_2, \ldots, b_m = e_m$, such that $b_1, b_2, b_3, \ldots, b_m \subset V\,[T_B\,(T)]$. Then by the definition of total block graph, $\{u_1, u_2, \ldots, u_{m-1}, u_m\} \cup \{b_1, b_2, \ldots, b_{m-1}, b_m\} \cup \{b_1u_1, b_2u_2, \ldots b_mu_{n-1}, b_mu_n\}$ form line set of $T_B\,(T)$ (see Figure 4).

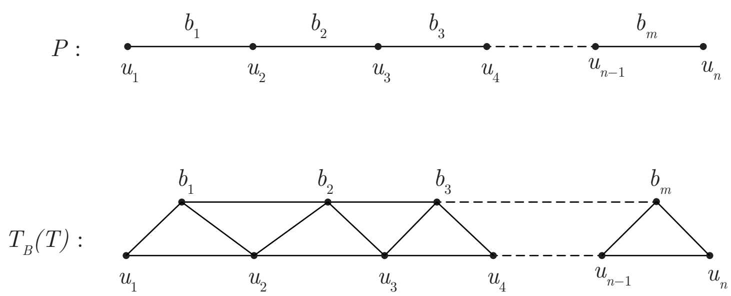


**Figure** 4

Now this path has exactly one pathos say $w$. In forming pathos entire total block graph of a path, the pathos $w$ becomes a point, then $V\,[P_{etb}(T)] = \{u_1, u_2, \cdots, u_n\} \cup \{b_1, b_2, \ldots, b_m\} \cup \{w\}$ and $w$ is adjacent to all the points $\{u_1, u_2, \cdots, u_n\}$ shown in the Figure 5.

In $P_{etb}(T)$, the hamiltonian cycle $w, u_1, b_1, u_2, b_2, u_2, u_3, b_3, \cdots, u_{n-1}, b_m, u_n, w$ exist. Clearly the pathos entire total block graph of a path is hamiltonian graph.

**Case** 2 Suppose $T$ is not a path. Then $T$ has at least one point with degree at least 3. Assume that $T$ has exactly one point $u$ such that degree $> 2$. Now we consider the following subcases of Case 2.

**Subcase** 2.1 Assume $T = K_{1,n}, n > 2$ and is odd. Then the number of paths of pathos are $\dfrac{n+1}{2}$. Let $V\,[T_B(T)] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots,\ b_{m-1}\}$. By the definition of pathos total block graph. By the definition $P_{etb}(T)$ $V\,[P_{etb}(T)] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots, b_{n-1}\} \cup \{p_1, p_2, \cdots, p_{n+1/2}\}$. Then there exists a cycle containing the points of By the definition of $P_{etb}(T)$ as

$p_1, u_1, b_1, b_2, u_3, p_2, u_2, b_3, u_4, \cdots, p_1$ and is a hamiltonian cycle. Hence $P_{etb}(T)$ is a hamiltonian.

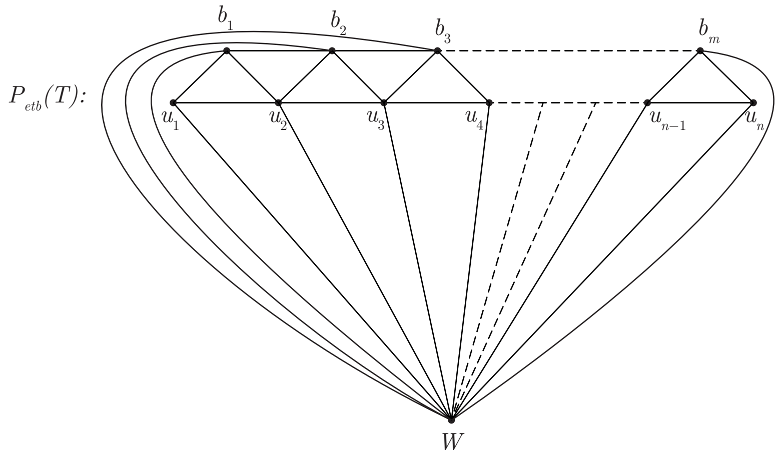


**Figure** 5

**Subcase** 2.2 Assume $T = K_{1,n}, n > 2$ and is even. Then the number of path of pathos are $\frac{n}{2}$, then $V[T_B(T)] = \{u_1, u_2, \cdots, u_n, b_1, b_2, \cdots, b_{n-1}\}$. By the definition of pathos entire total block graph $P_{etb}(T)$ of a tree $T$. $V[P_{etb}(T)] = \{u_1, u_2, \cdots, u_n, b_1, b_2, \cdots, b_{n-1}\} \cup \{p_1, p_2, \cdots, p_{n/2}\}$. Then there exist a cycle containing the points of $P_{etb}(T)$ as $p_1, u_1, b_1, b_2, u_3, p_2, u_4, b_3, b_4, \cdots, p_1$ and is a hamiltonian cycle. Hence $P_{etb}(T)$ is a hamiltonian.

Suppose $T$ is neither a path nor a star, then $T$ contains at least two points of degree $> 2$. Let $u_1, u_2, u_3, \cdots, u_n$ be the points of degree $\geq 2$ and $v_1, v_2, v_3, \cdots, v_m$ be the end points of $T$. Since end block is a line in $T$, and denoted as $b_1, b_2, \cdots, b_k$ ,then $V[T_B(T)] = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots v_m\} \cup \{b_1, b_2, \ldots, b_k\}$,and tree $T$ has $p_i$ pathos points, $i > 1$ and each pathos point is adjacent to the point of $T$ where the corresponding pathos lie on the points of $T$. Let $\{p_1, p_2, \cdots, p_i\}$be the pathos points of $T$. Then there exists a cycle $C$ containing all the points of $P_{etb}(T)$, as $p_1, v_1, b_1, v_2, p_2, u_1, b_3, u_2, p_3, v_3, b_4, v_{m-1}, b_{n-1}, b_n, v_m, \ldots, p_1$. Hence $P_{etb}(T)$ is a hamiltonian cycle. Clearly, $P_{etb}(T)$ is a hamiltonian graph. □

In the next theorem we characterize $P_{etb}(T)$ in terms of crossing number one.

**Theorem** 5.6 *For any non-trivial tree $T$, the pathos entire total block graph $P_{etb}(T)$ of a tree $T$ has crossing number one if and only if $\Delta(T) \leq 4$, and there exist a unique point in $T$ of degree 4.*

*Proof* Suppose $P_{etb}(T)$ has crossing number one. Then it is nonplanar. Then by Theorem 5.1, we have $\Delta(T) \leq 4$. We now consider the following cases.

**Case** 1 Assume $\Delta(T) = 5$. Then by Theorem [F], $T_B(T)$ is nonplanar with crossing number more than one. Since $T_B(T)$ is a subgraph of $P_{T_B}(T)$. Clearly $cr(P_{T_B}(T)) > 1$, a contradiction.

**Case** 2 Assume $\Delta(T) = 4$. Suppose $T$ has two points of degree 4. Then by Theorem F, $T_B(T)$ has crossing number at least two. But $T_B(T)$ is a subgraph of $P_{etb}(T)$. Hence $cr(P_{etb}(T)) > 1$, a contradiction.

Conversely, suppose $T$ satisfies the given condition and assume $T$ has a unique point $v$ of degree 4. The lines which are blocks in $T$ such that they are the points in $T_B(T)$. In $T_B(T)$, these block points and a point $v$ together forms an induced subgraph as $k_5$. In forming $P_{etb}(T)$, the pathos points are adjacent to at least two points of this induced subgraph. Hence in all these cases the $cr(P_{etb}(T)) = 1$. This completes the proof. □

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