

On a problem of F.Smarandache

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Abstract For any positive integer n , the famous Euler function $\phi(n)$ is defined as the number of all integers m with $1 \leq m \leq n$ such that $(m, n) = 1$. In his book "Only problems, not solutions" (see unsolved problem 52), Professor F.Smarandache asked us to find the smallest positive integer $k \equiv k(n)$, such that $\phi_k(n) = 1$, where $\phi_1(n) = \phi(n)$, $\phi_2(n) = \phi(\phi_1(n))$, \dots , and $\phi_k(n) = \phi(\phi_{k-1}(n))$. In this paper, we using the elementary method to study this problem, and prove that for any positive integer n , $k(n) = \min\{m : 2^m \geq n, m \in N\}$, where N denotes the set of all positive integers.

Keywords The Smarandache problem, Euler function, elementary method.

§1. Introduction and Results

For any positive integer n , the famous Euler function $\phi(n)$ is defined as the number of all integers m with $1 \leq m \leq n$ such that $(m, n) = 1$. In his book "Only problems, not solutions" (see unsolved problem 52), Professor F.Smarandache asked us to find the smallest positive integer $k \equiv k(n)$, such that $\phi_k(n) = 1$, where $\phi_1(n) = \phi(n)$, $\phi_2(n) = \phi(\phi_1(n))$, \dots , and $\phi_k(n) = \phi(\phi_{k-1}(n))$. That is, $k(n)$ is the smallest number of iteration k such that $\phi_k(n) = \phi(\phi_{k-1}(n)) = 1$. About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. The problem is interesting, because it can help us to know more properties of the Euler function. It is clear that $\phi(n) < n$, if $n > 1$. So $\phi_1(n), \phi_2(n), \phi_3(n), \dots, \phi_k(n)$ is a monotone decreasing sequence. Therefore, for any integer $n > 1$, there must exist a positive integer $k \equiv k(n)$ such that $\phi_k(n) = 1$. In this paper, we using the elementary method to study this problem, and find an exact function $k = k(n)$ such that for any integer $n > 1$, $\phi_k(n) = 1$. That is, we shall prove the following conclusion:

Theorem. For any positive integer $n > 1$, we define $k \equiv k(n) = \min\{m : 2^m \geq n, m \in N\}$, where N denotes the set of all positive integers. Then we have the identity

$$\phi_k(n) = \phi(\phi_{k-1}(n)) = \phi_{k-1}(\phi(n)) = 1,$$

where $\phi(n)$ is the Euler function.

Corollary. For any positive integer $n > 1$, Let $k \equiv k(n)$ be the smallest positive integer such that $\phi_k(n) = 1$. Then we have

$$k \equiv k(n) = \min\{m : 2^m \geq n, m \in N\}.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. For any integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. Then from the properties of the Euler function $\phi(n)$ we have

$$\phi(n) = p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_r^{\alpha_r-1}(p_r-1).$$

From this formula we know that if n be an even number, then $\phi(n) \leq \frac{n}{2}$. If $n > 1$ be an odd number, then $\phi(n) \leq n-1$, and $\phi(1) = 1$. So for any integer $n \geq 3$ and $k \equiv k(n) = \min\{m : 2^m \geq n, m \in N\}$, from the definition of $k = k(n)$ we have $\phi_1(n) = \phi(n) \leq n-1$, $\phi_2(n) = \phi(\phi(n)) \leq \frac{1}{2}\phi(n) \leq \frac{n-1}{2}$, \cdots , $\phi_k(n) \leq \frac{1}{2}\phi_{k-1}(n) \leq \cdots \leq \frac{n-1}{2^{k-1}} = \frac{2(n-1)}{2^k} \leq 2\left(1 - \frac{1}{n}\right)$. Since $\phi_k(n)$ is a positive integer and $1 \leq \phi_k(n) \leq 2\left(1 - \frac{1}{n}\right) < 2$, so we must have $\phi_k(n) = 1$.

For any positive integer n , Let $u \equiv u(n)$ be the smallest positive integer such that $\phi_u(n) = 1$. From the above we know that $\phi_k(n) = 1$, so $u(n) \leq k(n)$. On the other hand, let $n = 2^m$, where $m \geq 1$ be an integer. Then $\phi(n) = 2^{m-1}$, $\phi_2(n) = 2^{m-2}$, \cdots , $\phi_{m-1}(n) = 2$, $\phi_m(n) = 1$. So $u(n) = m = k(n)$. Let $n = 2^m + 1$ be a prime, then $\phi(n) = p-1 = 2^m$, $\phi_m(n) = 2$ and $\phi_{m+1}(n) = 1$. So $u(n) = m+1$. This time, we also have $k \equiv k(n) = \min\{s : 2^s \geq 2^m + 1, s \in N\} = m+1$. That is to say, there are infinite positive integers $n > 1$ such that $u(n) = k(n)$. Therefore, for any integer $n > 1$, $k \equiv k(n) = \min\{m : 2^m \geq n, m \in N\}$ be the smallest positive integer such that $\phi_k(n) = 1$. This completes the proof of our Theorem.

§3. Several similar problems

Now we consider the Dirichlet divisor function $d(n)$, the number of all positive divisors of n . For any integer $n \geq 3$, it is clear that $d(n) < n$. Let $d_1(n) = d(n)$, $d_2(n) = d(d(n))$, \cdots , $d_k(n) = d(d_{k-1}(n))$. So $\{d_1(n), d_2(n), \cdots, d_k(n), \cdots\}$ is also a monotone decreasing sequence. For any positive $n > 1$, let $k = k(n)$ be the smallest positive integer such that $d_k(n) = 2$. Whether there exists a simple arithmetical function $k = k(n)$ such that $d_k(n) = 2$ for all $n > 3$. This is an open problem.

For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. We define function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ and $\Omega(1) = 0$. Similarly, find the smallest positive integer $k = k(n)$ such that $\Omega_k(n) = 0$, where $\Omega_1(n) = \Omega(n)$, $\Omega_k(n) = \Omega(\Omega_{k-1}(n))$.

Here we can also give a simple arithmetical function for $k(n)$. Let $u_1 = 1$, $u_2 = 2^{u_1}$, \cdots , $u_{k+1} = 2^{u_k}$. It is clear that $\{u_k\}$ be a strictly monotone increasing sequence. Now we define $k \equiv k(n) = \min\{m : u_m \geq n, m \in N\}$, where N denotes the set of all positive integers. It is easy to prove that $\Omega_k(n) = 0$. On the other hand, for any positive integer $m > 1$, we have $\Omega(u_m) = u_{m-1}$, and $\Omega_m(u_m) = 0$. Therefore, for any integer $n > 1$, $k \equiv k(n) = \min\{m : u_m \geq n, m \in N\}$ be the smallest positive integer such that $\Omega_k(n) = 0$.

Whether there exists another more simple function $k(n)$ such that $\Omega_k(n) = 0$ is an unsolved problem.

Let $n > 1$ be an integer, and $\sigma(n)$ be the sum of all positive divisors of n . It is clear that $\sigma(n) > n$ for any $n > 1$. So if $n > 1$, then $\{\sigma_1(n), \sigma_2(n), \cdots, \sigma_k(n), \cdots\}$ must be a strictly

monotone increasing sequence, where $\sigma_1(n) = \sigma(n)$, and $\sigma_k(n) = \sigma(\sigma_{k-1}(n))$. Now let N be any fixed positive integer. For any integer $n \geq 2$, find the smallest positive integer $k = k(N)$ such that $\sigma_k(n) \geq N$.

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