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On a problem related to function $S(n)$

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. The main purpose of this paper is using the elementary method to study the number of all positive integer n such that $\frac{S(2)\cdot S(4)\cdot S(6)\cdot\ \cdots\ S(2n)}{S(1)\cdot S(3)\cdot S(5)\cdot\ \cdots\ S(2n-1)}$ is a positive integer.

Keywords F.Smarandache function $S(n)$, related problem, positive integer solution.

§1. Introduction and Result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. That is, $S(n) = \min\{m:\ m \in N,\ n|m!\}$, where N denotes the set of all positive integers. From the definition of $S(n)$, it is easy to see that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then we have

$$S(n) = \max_{1\leq i\leq k} \{S(p_i^{\alpha_i})\}.$$

It is clear that from this properties we can get the value of $S(n)$, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots\cdots$. About the arithmetical properties of $S(n)$, some authors had studied it, and obtained some interesting results. For example, Farris Mark and Mitchell Patrick [1] studied the bound of $S(n)$, and got the upper and lower bound estimates for $S(p^\alpha)$. They proved that

$$(p-1)\alpha + 1 \leq S(p^\alpha) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Lu Yaming [2] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

Jozsef Sandor [3] proved that for any positive integer $k \geq 2$, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ satisfying the inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ such that

$$S(m_1+m_2+\cdots+m_k) < S(m_1)+S(m_2)+\cdots+S(m_k).$$

In [4], Fu Jing proved a more general conclusion. That is, if the positive integer k and m satisfying one of the following conditions:

(a) $k > 2$ and $m \geq 1$ are odd numbers.

(b) $k \geq 5$ is odd, $m \geq 2$ is even.

(c) Any even number $k \geq 4$ and any positive integer m;

then the equation

$$m \cdot S(m_1+m_2+\cdots+m_k) = S(m_1)+S(m_2)+\cdots+S(m_k)$$

have infinite group positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

Xu Zhefeng [5] studied the value distribution properties of $S(n)$, and obtained a deeply result. That is, he proved the following Theorem:

Let $P(n)$ denotes the largest prime factor of n. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(S(n)-P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

On the other hand, in the manuscript "Problems lists for collective book on Smarandache notions", Kenichiro Kashihara proposed the following problem: Find all positive integer $n \in N$ such that

$$\frac{S(2)\cdot S(4)\cdot S(6)\cdot\ \cdots\ S(2n)}{S(1)\cdot S(3)\cdot S(5)\cdot\ \cdots\ S(2n-1)} \tag{1}$$

is an integer. About this problem, it seems that none had studied it yet, at least we have not seen related papers before. In this paper, we using the elementary method to study this problem, and solved it completely. We shall prove the following conclusion:

Theorem. For any positive integer n, the formula

$$\frac{S(2)\cdot S(4)\cdot S(6)\cdot\ \cdots\ S(2n)}{S(1)\cdot S(3)\cdot S(5)\cdot\ \cdots\ S(2n-1)}$$

is an integer if and only if $n = 1$.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem. First we need the following two simple lemmas.

Lemma 1. For any positive integer $n \geq 5$, there exists at least one prime P such that $P \in (n,\ 2n-1]$.

Proof. See Theorem 5.7.1 of reference [6].

Lemma 2. Let p be a prime. Then for any positive integer k, we have the estimate $S(p^k) \leq kp$. If $k \leq p$, then $S(p^k) = kp$.

Proof. See reference [1].

Now we use these two lemmas to complete the proof of our theorem. It is clear that if $n = 1$, then

$$\frac{S(2) \cdot S(4) \cdot S(6) \cdot \ \cdots \ S(2n)}{S(1) \cdot S(3) \cdot S(5) \cdot \ \cdots \ S(2n-1)}$$

is an integer. In fact, this time we have

$$\frac{S(2) \cdot S(4) \cdot S(6) \cdot \ \cdots \ S(2n)}{S(1) \cdot S(3) \cdot S(5) \cdot \ \cdots \ S(2n-1)} = \frac{S(2)}{S(1)} = 1.$$

If $n = 2$, we have $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, obviously formula (1) is not an integer.

Similarly, if $n = 3$ and 4, we have $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, obviously (1) is not also an integer.

Now we assume that $n > 4$. From Lemma 1 we know that there exists at least one prime $P \in (n,\ 2n-1]$ such that $S(P) = P$. For this prime P, we have $P \mid S(1) \cdot S(3) \cdot S(5) \cdot \cdots \ S(2n-1)$. But we can prove that

$$P \ \dagger \ S(2) \cdot S(4) \cdot S(6) \cdot \ \cdots \ S(2n).$$

Otherwise, there exists an integer k with $1 \leq k \leq n$ such that $P \mid S(2k)$. Let $S(2k) = \alpha P$. If $\alpha = 1$, then $S(2k) = P$ and $P \mid 2k$. So $P \mid k$. Therefore, $k \geq P \geq n+1$. This is a contradiction with $1 \leq k \leq n$. If $\alpha \geq 2$, then from Lemma 2 we have $2k \geq S(2k) = \alpha P \geq 2P \geq 2(n+1)$, or $k \geq P \geq n+1$, contradict with $1 \leq k \leq n$.

This completes the proof of our theorem.

References

[1] Farris Mark and Mitchell Patrick, Bounding the Smarandache function, Smarandache Notions Journal, **13**(2002), 37-42.

[2] Liu Yaming, On the solutions of an equation involving the Smarandache function, Scientia Magna, **2**(2006), No. 1, 76-79.

[3] Jozsef Sandor, On certain inequalities involving the Smarandache function, Scientia Magna, **2**(2006), No. 3, 78-80.

[4] Fu Jing, An equation involving the Smarandache function, Scientia Magna, **2**(2006), No. 4, 83-86.

[5] Xu Zhefeng，The value distribution property of the Smarandache function, Acta Mathematica Sinica (in Chinese), **49**(2006), No. 5, 1009-1012.

[6] L. Hua, Introduction to Number Theory, Springer-Verlag, New York, 1982.

[7] Kenichiro Kashihara, Comments and topics on Smarandache notions and problems, Erhus University Press, USA, 1996.