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The product of divisors minimum and maximum functions

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Abstract Let $T(n)$ denote the product of divisors of the positive integer n. We introduce and study some basic properties involving two functions, which are the minimum, resp. the maximum of certain integers connected with the divisors of T(n).

Keywords Arithmetic functions, product of divisors of an integer.

1. Let $T(n) = \prod_{i|n} i$ denote the product of all divisors of n. The product-of-divisors minimum, resp. maximum functions will be defined by

$$\mathcal{T}(n) = \min\{k \geq 1 : \ n|T(k)\} \tag{1}$$

and

$$\mathcal{T}_*(n) = \max\{k \geq 1 : \ T(k)|n\}. \tag{2}$$

There are particular cases of the functions F_f^A, G_g^A defined by

$$F_f^A(n) = \min\{k \in A : \ n|f(k)\}, \tag{3}$$

and its "dual"

$$G_g^A(n) = \max\{k \in A : \ g(k)|n\}, \tag{4}$$

where $A \subset \mathbb{N}^*$ is a given set, and $f, g : \mathbb{N}^* \to \mathbb{N}$ are given functions, introduced in [8] and [9]. For $A = \mathbb{N}^*$, $f(k) = g(k) = k!$ one obtains the Smarandache function $S(n)$, and its dual $S_*(n)$, given by

$$S(n) = \min\{k \geq 1 : \ n|k!\} \tag{5}$$

and

$$S_*(n) = \max\{k \geq 1 : \ k!|n\}. \tag{6}$$

The function $S_*(n)$ has been studied in [8], [9], [4], [1], [3]. For $A = \mathbb{N}^*$, $f(k) = g(k) = \varphi(k)$, one obtains the Euler minimum, resp. maximum functions

$$E(n) = \min\{k \geq 1 : \ n|\varphi(k)\} \tag{7}$$

studied in [6], [8], [13], resp., its dual

$$E_*(n) = \max\{k \geq 1 : \ \varphi(k)|n\}, \tag{8}$$

studied in [13].

For $A = \mathbb{N}^*$, $f(k) = g(k) = S(k)$ one has the Smarandache minimum and maximum functions

$$S_{min}(n) = \min\{k \geq 1 : \ n|S(k)\}, \tag{9}$$

$$S_{max}(n) = \max\{k \geq 1 : \ S(k)|n\}, \tag{10}$$

introduced, and studied in [15]. The divisor minimum function

$$D(n) = \min\{k \geq 1 : \ n|d(k)\} \tag{11}$$

(where $d(k)$ is the number of divisors of k) appears in [14], while the sum-of-divisors minimum and maximum functions

$$\Sigma(n) = \min\{k \geq 1 : \ n|\sigma(k)\} \tag{12}$$

$$\Sigma_*(n) = \max\{k \geq 1 : \ \sigma(k)|n\} \tag{13}$$

have been recently studied in [16].

For functions $Q(n), Q_1(n)$ obtained from (3) for $f(k) = k!$ and $A =$ set of perfect squares, resp. $A =$ set of squarefree numbers, see [10].

2. The aim of this note is to study some properties of the functions $\mathcal{T}(n)$ and $\mathcal{T}_*(n)$ given by (1) and (2). We note that properties of $T(n)$ in connection with "multiplicatively perfect numbers" have been introduced in [11]. For other asymptotic properties of $T(n)$, see [7]. For divisibility properties of $T(\sigma(n))$ with $T(n)$, see [5]. For asymptotic results of sums of type $\sum\limits_{n \leq x} \frac{1}{T(n)}$, see [17].

A divisor i of n is called "unitary" if $\left(i, \frac{n}{i}\right) = 1$. Let $T^*(n)$ be the product of unitary divisors of n. For similar results to [11] for $T^*(n)$, or $T^{**}(n)$ (i.e. the product of "bi-unitary" divisors of n), see [2]. The product of "exponential" divisors $T_e(n)$ is introduced in paper [12]. Clearly, one can introduce functions of type (1) and (2) for $T(n)$ replaced with one of the above functions $T^*(n), T^{**}, T_e(n)$, but these functions will be studied in another paper.

3. The following auxiliary result will be important in what follows.

Lemma 1.

$$T(n) = n^{d(n)/2}, \tag{14}$$

where $d(n)$ is the number of divisors of n.

Proof. This is well-known, see e.g. [11].

Lemma 2.

$$T(a)|T(b), \quad \text{if } a|b. \tag{15}$$

Proof. If $a|b$, then for any $d|a$ one has $d|b$, so $T(a)|T(b)$. Reciprocally, if $T(a)|T(b)$, let $\gamma_p(a)$ be the exponent of the prime in a. Clearly, if $p|a$, then $p|b$, otherwise $T(a)|T(b)$ is impossible. If $p^{\gamma_p(b)}\|b$, then we must have $\gamma_p(a) \leq \gamma_p(b)$. Writing this fact for all prime divisors of a, we get $a|b$.

Theorem 1. If n is squarefree, then

$$\mathcal{T}(n) = n. \tag{16}$$

Proof. Let $n = p_1p_2 \dots p_r$, where p_i $(i = \overline{1,r})$ are distinct primes. The relation $p_1p_2 \dots p_r | T(k)$ gives $p_i | T(k)$, so there is a $d|k$, so that $p_i|d$. But then $p_i|k$ for all $i = \overline{1,r}$, thus $p_1p_2 \dots p_r = n|k$. Since $p_1p_2 \dots p_k | T(p_1p_2 \dots p_k)$, the least k is exactly $p_1p_2 \dots p_r$, proving (16).

Remark. Thus, if p is a prime, $\mathcal{T}(p) = p$; if $p < q$ are primes, then $\mathcal{T}(pq) = pq$, etc.

Theorem 2. If $a|b$, $a \neq b$ and b is squarefree, then

$$\mathcal{T}(ab) = b. \tag{17}$$

Proof. If $a|b$, $a \neq b$, then clearly $T(b) = \prod_{d|b} d$ is divisible by ab, so $\mathcal{T}(ab) \leq b$. Reciprocally, if $ab|T(k)$, let $p|b$ a prime divisor of b. Then $p|T(k)$, so (see the proof of Theorem 1) $p|k$. But b being squarefree (i.e. a product of distinct primes), this implies $b|k$. The least such k is clearly $k = b$.

For example, $\mathcal{T}(12) = \mathcal{T}(2 \cdot 6) = 6$, $\mathcal{T}(18) = \mathcal{T}(3 \cdot 6) = 6$, $\mathcal{T}(20) = \mathcal{T}(2 \cdot 10) = 10$.

Theorem 3. $\mathcal{T}(T(n)) = n$ for all $n \geq 1$. (18)

Proof. Let $T(n)|T(k)$. Then by (15) one can write $n|k$. The least k with this property is $k = n$, proving relation (18).

Theorem 4. Let p_i $(i = \overline{1,r})$ be distinct primes, and $\alpha_i \geq 1$ positive integers. Then

$$\max\left\{\mathcal{T}\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) : \ i = \overline{1,r}\right\} \leq \mathcal{T}\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq$$

$$\leq l.c.m.[\mathcal{T}(p_1^{\alpha_1}), \dots, \mathcal{T}(p_r^{\alpha_r})]. \tag{19}$$

Proof. In [13] it is proved that for $A = \mathbb{N}^*$, and any function f such that $F_f^{\mathbb{N}^*}(n) = F_f(n)$ is well defined, one has

$$\max\{F_f(p_i^{\alpha_i}) : \ i = \overline{1,r}\} \leq F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right). \tag{20}$$

On the other hand, if f satisfies the property

$$a|b \Longrightarrow f(a)|f(b)(a, b \geq 1), \tag{21}$$

then

$$F_f\left(\prod_{i=1}^{r} p_i^{\alpha_i}\right) \leq l.c.m.[F_f(p_1^{\alpha_1}), \dots, F_f(p_r^{\alpha_r})]. \tag{22}$$

By Lemma 2, (21) is true for $f(a) = T(a)$, and by using (20), (22), relation (19) follows.

Theorem 5.

$$\mathcal{T}(2^n) = 2^{\alpha}, \tag{23}$$

where α is the least positive integer such that

$$\frac{\alpha(\alpha+1)}{2} \geq n. \tag{24}$$

Proof. By (14), $2^n|T(k)$ iff $2^n|k^{d(k)/2}$. Let $k = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, when $d(k) = (\alpha_1+1)\dots(\alpha_r+1)$. Since $2^{2n}|k^{d(k)} = p_1^{\alpha_1(\alpha_1+1)\dots(\alpha_r+1)} \dots p_r^{\alpha_r(\alpha_1+1)\dots(\alpha_r+1)}$ (let $p_1 < p_2 < \dots < p_r$), clearly $p_1 = 2$

and the least k is when $\alpha_2 = \cdots = \alpha_r = 0$ and α_1 is the least positive integer with $2n \leq \alpha_1(\alpha_1+1)$. This proves (23), with (24).

For example, $\mathcal{T}(2^2) = 4$, since $\alpha = 2$, $\mathcal{T}(2^3) = 4$ again, $\mathcal{T}(2^4) = 8$ since $\alpha = 3$, etc.

For odd prime powers, the things are more complicated. For example, for 3^n one has:

Theorem 6.

$$\mathcal{T}(3^n) = \min\{3^{\alpha_1}, 2\cdot 3^{\alpha_2}\}, \tag{25}$$

where α_1 is the least positive integer such that $\frac{\alpha_1(\alpha_1+1)}{2} \geq n$, and α_2 is the least positive integer such that $\alpha_2(\alpha_2+1) \geq n$.

Proof. As in the proof of Theorem 5,

$$3^{2n} | p_1^{\alpha_1(\alpha_1+1)\ldots(\alpha_r+1)} \cdot p_2^{\alpha_2(\alpha_1+1)\ldots(\alpha_1+1)} \ldots p_r^{\alpha_r(\alpha_1+1)\ldots(\alpha_r+1)},$$

where $p_1 < p_2 < \cdots < p_r$, so we can distinguish two cases:

a) $p_1 = 2,\ p_2 = 3,\ p_3 \geq 5$;

b) $p_1 = 3,\ p_2 \geq 5$.

Then $k = 2^{\alpha_1}\cdot 3^{\alpha_2}\ldots p_r^{\alpha_r} \geq 2^{\alpha_1}\cdot 3^{\alpha_2}$ in case a), and $k \geq 3^{\alpha_1}$ in case b). So for the least k we must have $\alpha_2(\alpha_1+1)(\alpha_2+1) \geq 2n$ with $\alpha_1 = 1$ in case a), and $\alpha_1(\alpha_1+1) \geq 2n$ in case b). Therefore $\frac{\alpha_1(\alpha_1+1)}{2} \geq n$ and $\alpha_2(\alpha_2+1) \geq n$, and we must select k with the least of 3^{α_1} and $2^1 \cdot 3^{\alpha_2}$, so Theorem 6 follows.

For example, $\mathcal{T}(3^2) = 6$ since for $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, and $\min\{2\cdot 3^1, 3^2\} = 6$; $\mathcal{T}(3^3) = 9$ since for $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 2$ and $\min\{2\cdot 3^2, 3^2\} = 9$.

Theorem 7. Let $f : [1,\infty) \to [0,\infty)$ be given by $f(x) = \sqrt{x}\log x$. Then

$$f^{-1}(\log n) < \mathcal{T}(n) \leq n, \tag{26}$$

for all $n \geq 1$, where f^{-1} denotes the inverse function of f.

Proof. Since $n|T(n)$, the right side of (26) follows by definition (1) of $\mathcal{T}(n)$. On the other hand, by the known inequality $d(k) < 2\sqrt{k}$, and Lemma 1 (see (14)) we get $T(k) < k^{\sqrt{k}}$, so $\log T(k) < \sqrt{k}\log k = f(k)$. Since $n|T(k)$ implies $n \leq T(k)$, so $\log n \leq \log T(k) < f(k)$, and the function f being strictly increasing and continuous, by the bijectivity of f, the left side of (26) follows.

4. The function $\mathcal{T}_*(n)$ given by (2) differs in many aspects from $\mathcal{T}(n)$. The first such property is:

Theorem 8. $\mathcal{T}_*(n) \leq n$ for all n, with equality only if $n = 1$ or $n =$ prime.

Proof. If $T(k)|n$, then $T(k) \leq n$. But $T(k) \geq k$, so $k \leq n$, and the inequality follows.

Let us now suppose that for $n > 1$, $\mathcal{T}_*(n) = n$. Then $T(n)|n$, by definition 2. On the other hand, clearly $n|T(n)$, so $T(n) = n$. This is possible only when $n =$ prime.

Remark. Therefore the equality

$$\mathcal{T}_*(n) = n (n > 1)$$

is a characterization of the prime numbers.

Lemma 3. Let $p_1, \ldots, p_r$ be given distinct primes $(r \geq 1)$. Then the equation

$$T(k) = p_1p_2\ldots p_r$$

is solvable if $r = 1$.

Proof. Since $p_i | T(k)$, we get $p_i | k$ for all $i = \overline{1, r}$. Thus $p_1 \dots p_r | k$, and Lemma 2 implies $T(p_1 \dots p_r) | T(k) = p_1 \dots p_r$. Since $p_1 \dots p_r | T(p_1 \dots p_r)$, we have $T(p_1 \dots p_r) = p_1 \dots p_r$, which by Theorem 8 is possible only if $r = 1$.

Theorem 9. Let $P(n)$ denote the greatest prime factor of $n > 1$. If n is squarefree, then

$$\mathcal{T}_*(n) = P(n). \tag{27}$$

Proof. Let $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots < p_r$. If $T(k) | (p_1 \dots p_r)$, then clearly $T(k) \in \{1, p_1, \dots, p_r, p_1 p_2, \dots, p_1 p_2 \dots p_r\}$. By Lemma 3 we cannot have

$$T(k) \in \{p_1 p_2, \dots, p_1 p_2 \dots p_r\},$$

so $T(k) \in \{1, p_1, \dots, p_r\}$, when $k \in \{1, p_1, \dots, p_r\}$. The greatest k is $p_r = P(n)$.

Remark. Therefore $\mathcal{T}_*(pq) = q$ for $p < q$. For example, $\mathcal{T}_*(2 \cdot 7) = 7$, $\mathcal{T}_*(3 \cdot 5) = 5$, $\mathcal{T}_*(3 \cdot 7) = 7$, $\mathcal{T}_*(2 \cdot 11) = 11$, etc.

Theorem 10.

$$\mathcal{T}_*(p^n) = p^\alpha (p = \text{prime}), \tag{28}$$

where α is the greatest integer with the property

$$\frac{\alpha(\alpha+1)}{2} \leq n. \tag{29}$$

Proof. If $T(k) | p^n$, then $T(k) = p^m$ for $m \leq n$. Let q be a prime divisor of k. Then $q = T(q) | T(k) = 2^m$ implies $q = p$, so $k = p^\alpha$. But then $T(k) = p^{\alpha(\alpha+1)/2}$ with α the greatest number such that $\alpha(\alpha+1)/2 \leq n$, which finishes the proof of (28).

For example, $\mathcal{T}_*(4) = 2$, since $\frac{\alpha(\alpha+1)}{2} \leq 2$ gives $\alpha_{max} = 1$.

$\mathcal{T}_*(16) = 4$, since $\frac{\alpha(\alpha+1)}{2} \leq 4$ is satisfied with $\alpha_{max} = 2$.

$\mathcal{T}_*(9) = 3$, and $\mathcal{T}_*(27) = 9$ since $\frac{\alpha(\alpha+1)}{2} \leq 3$ with $\alpha_{max} = 2$.

Theorem 11. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^2 q) = \max\{p, q\}. \tag{30}$$

Proof. If $T(k) | p^2 q$, then $T(k) \in \{1, p, q, p^2, pq, p^2 q\}$. The equations $T(k) = p^2$, $T(k) = pq$, $T(k) = p^2 q$ are impossible. For example, for the first equation, this can be proved as follows. By $p | T(k)$ one has $p | k$, so $k = pm$. Then $p(pm)$ are in $T(k)$, so $m = 1$. But then $T(k) = p \neq p^2$. For the last equation, $k = (pq)m$ and $pqm(pm)(qm)(pqm)$ are in $T(k)$, which is impossible.

Theorem 12. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^3 q) = \max\{p^2, q\}. \tag{31}$$

Proof. As above, $T(k) \in \{1, p, q, pq, p^2 q, p^3 q, p^2, p^3\}$ and $T(k) \in \{pq, p^2 q, p^3 q, p^2\}$ are impossible. But $T(k) = p^3$ by Lemma 1 gives $k^{d(k)} = p^6$, so $k = p^m$, when $d(k) = m + 1$. This gives $m(m+1) = 6$, so $m = 2$. Thus $k = p^2$. Since $p < p^2$ the result follows.

Remark. The equation

$$T(k) = p^s \tag{32}$$

can be solved only if $k^{d(k)} = p^{2s}$, so $k = p^m$ and we get $m(m+1) = 2s$. Therefore $k = p^m$, with $m(m+1) = 2s$, if this is solvable. If s is not a **triangular number**, this is impossible.

Theorem 13. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^s q) = \begin{cases} \max\{p, q\}, & \text{if } s \text{ is not a triangular number,} \\ \max\{p^n, q\}, & \text{if } s = \frac{m(m+1)}{2}. \end{cases}$$

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