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Some Properties of Birings

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Abstract: Let R be any ring and let $S = R_1 \cup R_2$ be the union of any two subrings of R. Since in general S is not a subring of R but R_1 and R_2 are algebraic structures on their own under the binary operations inherited from the parent ring R, S is recognized as a bialgebraic structure and it is called a biring. The purpose of this paper is to present some properties of such bialgebraic structures.

Key Words: Biring, bi-subring, bi-ideal, bi-field and bidomain

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§1. Introduction

Generally speaking, the unions of any two subgroups of a group, subgroupoids of a groupoid, subsemigroups of a semigroup, submonoids of a monoid, subloops of a loop, subsemirings of a semiring, subfields of a field and subspaces of a vector space do not form any nice algebraic structures other than ordinary sets. Similarly, if S_1 and S_2 are any two subrings of a ring R, I_1 and I_2 any two ideals of R, the unions $S = S_1 \cup S_2$ and $I = I_1 \cup I_2$ generally are not subrings and ideals of R, respectively [2]. However, the concept of bialgebraic structures recently introduced by Vasantha Kandasamy [9] recognises the union $S = S_1 \cup S_2$ as a biring and $I = I_1 \cup I_2$ as a bi-ideal. One of the major advantages of bialgebraic structures is the exhibition of distinct algebraic properties totally different from those inherited from the parent structures. The concept of birings was introduced and studied in [9]. Other related bialgebraic structures introduced in [9] included binear-rings, bisemi-rings, biseminear-rings and group birings. Agboola and Akinola in [1] studied bicoset of a bivector space. Also, we refer the readers to [3-7]. In this paper, we will present and study some properties of birings.

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§2. Definitions and Elementary Properties of Birings

Definition 2.1 *Let R_1 and R_2 be any two proper subsets of a non-empty set R. Then, $R = R_1 \cup R_2$ is said to be a biring if the following conditions hold:*

(1) *R_1 is a ring;*

(2) *R_2 is a ring.*

Definition 2.2 *A biring $R = R_1 \cup R_2$ is said to be commutative if R_1 and R_2 are commutative rings. $R = R_1 \cup R_2$ is said to be a non-commutative biring if R_1 is non-commutative or R_2 is non-commutative.*

Definition 2.3 *A biring $R = R_1 \cup R_2$ is said to have a zero element if R_1 and R_2 have different zero elements. The zero element 0 is written $0_1 \cup 0_2$ (notation is not set theoretic union) where $0_i, i = 1, 2$ are the zero elements of R_i. If R_1 and R_2 have the same zero element, we say that the biring $R = R_1 \cup R_2$ has a mono-zero element.*

Definition 2.4 *A biring $R = R_1 \cup R_2$ is said to have a unit if R_1 and R_2 have different units. The unit element u is written $u_1 \cup u_2$, where $u_i, i = 1, 2$ are the units of R_i. If R_1 and R_2 have the same unit, we say that the biring $R = R_1 \cup R_2$ has a mono-unit.*

Definition 2.5 *A biring $R = R_1 \cup R_2$ is said to be finite if it has a finite number of elements. Otherwise, R is said to be an infinite biring. If R is finite, the order of R is denoted by $o(R)$.*

Example 1 Let $R = \{0, 2, 4, 6, 7, 8, 10, 12\}$ be a subset of $\mathcal{Z}_{14}$. It is clear that $(R, +, \cdot)$ is not a ring but then, $R_1 = \{0, 7\}$ and $R_2 = \{0, 2, 4, 6, 8, 10, 12\}$ are rings so that $R = R_1 \cup R_2$ is a finite commutative biring.

Definition 2.6 *Let $R = R_1 \cup R_2$ be a biring. A non-empty subset S of R is said to be a sub-biring of R if $S = S_1 \cup S_2$ and S itself is a biring and $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$.*

Theorem 2.7 *Let $R = R_1 \cup R_2$ be a biring. A non-empty subset $S = S_1 \cup S_2$ of R is a sub-biring of R if and only if $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$ are subrings of R_1 and R_2, respectively.*

Definition 2.8 *Let $R = R_1 \cup R_2$ be a biring and let x be a non-zero element of R. Then,*

(1) *x is a zero-divisor in R if there exists a non-zero element y in R such that $xy = 0$;*

(2) *x is an idempotent in R if $x^2 = x$;*

(3) *x is nilpotent in R if $x^n = 0$ for some $n > 0$.*

Example 2 Consider the biring $R = R_1 \cup R_2$, where $R_1 = \mathcal{Z}$ and $R_2 = \{0, 2, 4, 6\}$ a subset of $\mathcal{Z}_8$.

(1) If $S_1 = 4\mathcal{Z}$ and $S_2 = \{0, 4\}$, then S_1 is a subring of R_1 and S_2 is a subring of R_2. Thus, $S = S_1 \cup S_2$ is a bi-subring of R since $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$.

(2) If $S_1 = 3\mathcal{Z}$ and $S_2 = \{0, 4\}$, then $S = S_1 \cup S_2$ is a biring but not a bi-subring of R because $S_1 \neq S \cap R_1$ and $S_2 \neq S \cap R_2$. This can only happen in a biring structure.

Theorem 2.9 *Let $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ be any two birings and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be sub-birings of R and S, respectively. Then,*

(1) $R \times S = (R_1 \times S_1) \cup (R_2 \times S_2)$ *is a biring;*

(2) $I \times J = (I_1 \times J_1) \cup (I_2 \times J_2)$ *is a sub-biring of* $R \times S$.

Definition 2.10 *Let* $R = R_1 \cup R_2$ *be a biring and let* I *be a non-empty subset of* R.

(1) I *is a right bi-ideal of* R *if* $I = I_1 \cup I_2$*, where* I_1 *is a right ideal of* R_1 *and* I_2 *is a right ideal of* R_2*;*

(2) I *is a left bi-ideal of* R *if* $I = I_1 \cup I_2$*, where* I_1 *is a left ideal of* R_1 *and* I_2 *is a left ideal of* R_2*;*

(3) $I = I_1 \cup I_2$ *is a bi-ideal of* R *if* I_1 *is an ideal of* R_1 *and* I_2 *is an ideal of* R_2.

Definition 2.11 *Let* $R = R_1 \cup R_2$ *be a biring and let* I *be a non-empty subset of* R*. Then,* $I = I_1 \cup I_2$ *is a mixed bi-ideal of* R *if* I_1 *is a right (left) ideal of* R_1 *and* I_2 *is a left (right) ideal of* R_2.

Theorem 2.12 *Let* $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ *and* $K = K_1 \cup K_2$ *be left (right) bi-ideals of a biring* $R = R_1 \cup R_2$*. Then,*

(1) $IJ = (I_1J_1) \cup (I_2J_2)$ *is a left(right) bi-ideal of* R*;*

(2) $I \cap J = (I_1 \cap J_1) \cup (I_2 \cap J_2)$ *is a left(right) bi-ideal of* R*;*

(3) $I + J = (I_1 + J_1) \cup (I_2 + J_2)$ *is a left(right) bi-ideal of* R*;*

(4) $I \times J = (I_1 \times J_1) \cup (I_2 \times J_2)$ *is a left(right) bi-ideal of* R*;*

(5) $(IJ)K = \Big((I_1J_1)K_1\Big) \cup \Big((I_2J_2)K_2\Big) = I(JK) = \Big(I_1(J_1K_1)\Big) \cup \Big(I_2(J_2K_2)\Big)$*;*

(6) $I(J+K) = \Big(I_1(J_1+K_1)\Big) \cup \Big(I_2(J_2+K_2)\Big) = IJ+IK = (I_1J_1+I_1K_1) \cup (I_2J_2+I_2K_2)$*;*

(7) $(J+K)I = \Big((J_1+K_1)I_1\Big) \cup \Big((J_2+K_2)I_2\Big) = JI+KI = (J_1I_1+K_1I_1) \cup (J_2I_2+K_2I_2)$.

Example 3 Let R be the collection of all 2×2 upper triangular and lower triangular matrices over a field F and let

$$R_1 = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in F \right\},$$

$$R_2 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F \right\},$$

$$I_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in F \right\},$$

$$I_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} : a \in F \right\}.$$

Then, $R = R_1 \cup R_2$ is a non-commutative biring with a mono-unit $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I = I_1 \cup I_2$ is a right bi-ideal of $R = R_1 \cup R_2$.

Definition 2.13 *Let* $R = R_1 \cup R_2$ *and* $S = S_1 \cup S_2$ *be any two birings. The mapping* $\phi : R \to S$

is called a biring homomorphism if $\phi = \phi_1 \cup \phi_2$ *and* $\phi_1 : R_1 \to S_1$ *and* $\phi_2 : R_2 \to S_2$ *are ring homomorphisms. If* $\phi_1 : R_1 \to S_1$ *and* $\phi_2 : R_2 \to S_2$ *are ring isomorphisms, then* $\phi = \phi_1 \cup \phi_2$ *is a biring isomorphism and we write* $R = R_1 \cup R_2 \cong S = S_1 \cup S_2$. *The image of* ϕ *denoted by* $Im\phi = Im\phi_1 \cup Im\phi_2 = \{y_1 \in S_1, y_2 \in S_2 : y_1 = \phi_1(x_1), y_2 = \phi_2(x_2) \text{ for some } x_1 \in R_1, x_2 \in R_2\}$. *The kernel of* ϕ *denoted by*

$$Ker\phi = Ker\phi_1 \cup Ker\phi_2 = \{a_1 \in R_1, a_2 \in R_2 : \phi_1(a_1) = 0 \text{ and } \phi_2(a_2) = 0\}.$$

Theorem 2.14 *Let* $R = R_1 \cup R_2$ *and* $S = S_1 \cup S_2$ *be any two birings and let* $\phi = \phi_1 \cup \phi_2 : R \to S$ *be a biring homomorphism. Then,*

(1) $Im\phi$ *is a sub-biring of the biring S;*

(2) $Ker\phi$ *is a bi-ideal of the biring R;*

(3) $Ker\phi = \{0\}$ *if and only if* $\phi_i, i = 1, 2$ *are injective.*

Proof (1) It is clear that $Im\phi = Im\phi_1 \cup Im\phi_2$, where $\phi_1 : R_1 \to S_1$ and $\phi_2 : R_2 \to S_2$ are ring homomorphisms, is not an empty set. Since $Im\phi_1$ is a subring of S_1 and $Im\phi_2$ is a subring of S_2, it follows that $Im\phi = Im\phi_1 \cup Im\phi_2$ is a biring. Lastly, it can easily be shown that $Im\phi \cap S_1 = Im\phi_1$,$Im\phi \cap S_2 = Im\phi_2$ and consequently, $Im\phi = Im\phi_1 \cup Im\phi_2$ is a sub-biring of the biring $S = S_1 \cup S_2$.

(2) The proof is similar to (1).

(3) It is clear. □

Let $I = I_1 \cup I_2$ be a left bi-ideal of a biring $R = R_1 \cup R_2$. We know that R_1/I_1 and R_2/I_2 are factor rings and therefore $(R_1/I_1) \cup (R_2/I_2)$ is a biring called *factor-biring*. Since $\phi_1 : R_1 \to R_1/I_1$ and $\phi_2 : R_2 \to R_2/I_2$ are natural homomorphisms with kernels I_1 and I_2, respectively, it follows that $\phi_1 \cup \phi_2 = \phi : R \to R/I$ is a natural biring homomorphism whose kernel is $Ker\phi = I_1 \cup I_2$.

Theorem 2.15(First Isomorphism Theorem) *Let* $R = R_1 \cup R_2$ *and* $S = S_1 \cup S_2$ *be any two birings and let* $\phi_1 \cup \phi_2 = \phi : R \to S$ *be a biring homomorphism with kernel* $K = Ker\phi = Ker\phi_1 \cup Ker\phi_2$. *Then,* $R/K \cong Im\phi$.

Proof Suppose that $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ are birings and suppose that $\phi_1 \cup \phi_2 = \phi : R \to S$ is a biring homomorphism with kernel $K = Ker\phi = Ker\phi_1 \cup Ker\phi_2$. Then, K is a bi-ideal of R, $Im\phi = Im\phi_1 \cup Im\phi_2$ is a bi-subring of S and $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2)$ is a biring. From the classical rings (first isomorphism theorem), we have $R_i/Ker\phi_i \cong Im\phi_i, i = 1, 2$ and therefore, $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2) \cong Im\phi = Im\phi_1 \cup Im\phi_2$. □

Theorem 2.16(Second Isomorphism Theorem) *Let* $R = R_1 \cup R_2$ *be a biring. If* $S = S_1 \cup S_2$ *is a sub-biring of R and* $I = I_1 \cup I_2$ *is a bi-ideal of R, then*

(1) $S + I$ *is a sub-biring of R;*

(2) I *is a bi-ideal of* $S + I$*;*

(3) $S \cap I$ *is a bi-ideal of S;*

(4) $(S + I)/I \cong S/(S \cap I)$.

Proof Suppose that $R = R_1 \cup R_2$ is a biring, $S = S_1 \cup S_2$ a sub-biring and $I = I_1 \cup I_2$ a bi-ideal of R.

(1) $S + I = (S_1 + I_1) \cup (S_2 + I_2)$ is a biring since $S_i + I_i$ are subrings of R_i, where $i = 1, 2$. Now, $R_1 \cap (S + I) = \Big(R_1 \cap (S_1 + I_1)\Big) \cup \Big(R_1 \cap (S_2 + I_2)\Big) = S_1 + I_1$. Similarly, we have $R_2 \cap (S + I) = S_2 + I_2$. Thus, $S + I$ is a sub-biring of R.

(2) and (3) are clear.

(4) It is clear that $(S+I)/I = \Big((S_1+I_1)/I_1\Big) \cup \Big((S_2+I_2)/I_2\Big)$ is a biring since $(S_1+I_1)/I_1$ and $(S_2 + I_2)/I_2$ are rings. Similarly, $S/(S \cap I) = \Big(S_1/(S_1 \cap I_1)\Big) \cup \Big(S_2/(S_2 \cap I_2)\Big)$ is a biring. Consider the mapping $\phi = \phi_1 \cup \phi_2 : S_1 \cup S_2 \to \Big((S_1 + I_1)/I_1\Big) \cup \Big((S_2 + I_2)/I_2\Big)$. It is clear that ϕ is a biring homomorphism since $\phi_i : S_i \to (S_i + I_i)/I_i, i = 1, 2$ are ring homomorphisms. Also, since $Ker\phi_i = S_i \cap I_i, i = 1, 2$, it follows that $Ker\phi = (S_1 \cap I_1) \cup (S_2 \cap I_2)$. The required result follows from the first isomorphism theorem. □

Theorem 2.17(Third Isomorphism Theorem) *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be two bi-ideals of R such that $J_i \subseteq I_i, i = 1, 2$. Then,*

(1) *I/J is a bi-ideal of R/J;*

(2) *$R/I \cong (R/J)/(I/J)$.*

Proof Suppose that $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ are two bi-ideals of the biring $R = R_1 \cup R_2$ such that $J_i \subseteq I_i, i = 1, 2$.

(1) It is clear that $R/J = (R_1/J_1) \cup (R_2/J_2)$ and $I/J = (I_1/J_1) \cup (I_2/J_2)$ are birings. Now, $(R_1/J_1) \cap \Big((I_1/J_1) \cup (I_2/J_2)\Big) = \Big((R_1/J_1) \cap (I_1/J_1)\Big) \cup \Big((R_1/J_1) \cap (I_2/J_2)\Big) = I_1/J_1$ (since $J_i \subseteq I_i \subseteq R_i, i = 1, 2$). Similarly, $(R_2/J_2) \cap \Big((I_1/J_1) \cup (I_2/J_2)\Big) = I_2/J_2$. Consequently, I/J is a sub-biring of R/J and in fact a bi-ideal.

(2) Let us consider the mapping $\phi = \phi_1 \cup \phi_2 : (R_1/J_1) \cup (R_2/J_2) \to (R_1/I_1) \cup (R_2/I_2)$. Since $\phi_i : R_i/J_i \to R_i/I_i, i = 1, 2$ are ring homomorphisms with $Ker\phi_i = I_i/J_i$, it follows that $\phi = \phi_1 \cup \phi_2$ is a biring homomorphism and $Ker\phi = Ker\phi_1 \cup Ker\phi_2 = (I_1/J_1) \cup (I_2/J_2)$. Applying the first isomorphism theorem, we have $\Big((R_1/J_1)/(I_1/J_1)\Big) \cup \Big((R_2/J_2)/(I_2/J_2)\Big) \cong (R_1/I_1) \cup /(R_2/I_2)$. □

Definition 2.18 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. Then,*

(1) *I is said to be a principal bi-ideal of R if I_1 is a principal ideal of R_1 and I_2 is a principal ideal of R_2;*

(2) *I is said to be a maximal (minimal) bi-ideal of R if I_1 is a maximal (minimal) ideal of R_1 and I_2 is a maximal (minimal) ideal of R_2;*

(3) *I is said to be a primary bi-ideal of R if I_1 is a primary ideal of R_1 and I_2 is a primary ideal of R_2;*

(4) *I is said to be a prime bi-ideal of R if I_1 is a prime ideal of R_1 and I_2 is a prime ideal of R_2.*

Example 4 Let $R = R_1 \cup R_2$ be a biring, where $R_1 = \mathcal{Z}$, the ring of integers and $R_2 = \mathcal{R}[x]$, the ring of polynomials over $\mathcal{R}$. Let $I_1 = (2)$ and $I_2 = (x^2 + 1)$. Then, $I = I_1 \cup I_2$ is a principal bi-ideal of R.

Definition 2.19 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. Then, I is said to be a quasi maximal (minimal) bi-ideal of R if I_1 or I_2 is a maximal (minimal) ideal.*

Definition 2.20 *Let $R = R_1 \cup R_2$ be a biring. Then, R is said to be a simple biring if R has no non-trivial bi-ideals.*

Theorem 2.21 *Let $\phi = \phi_1 \cup \phi_2 : R \to S$ be a biring homomorphism. If $J = J_1 \cup J_2$ is a prime bi-ideal of S, then $\phi^{-1}(J)$ is a prime bi-ideal of R.*

Proof Suppose that $J = J_1 \cup J_2$ is a prime bi-ideal of S. Then, $J_i, i = 1, 2$ are prime ideals of S_i. Since $\phi^{-1}(J_i), i = 1, 2$ are prime ideals of R_i, we have $I = \phi^{-1}(J_1) \cup \phi^{-1}(J_2)$ to be a prime bi-ideal of R. □

Definition 2.22 *Let $R = R_1 \cup R_2$ be a commutative biring. Then,*

(1) *R is said to be a bidomain if R_1 and R_2 are integral domains;*

(2) *R is said to be a pseudo bidomain if R_1 and R_2 are integral domains but R has zero divisors;*

(3) *R is said to be a bifield if R_1 and R_2 are fields. If R is finite, we call R a finite bifield. R is said to be a bifield of finite characteristic if the characteristic of both R_1 and R_2 are finite. We call R a bifield of characteristic zero if the characteristic of both R_1 and R_2 is zero. No characteristic is associated with R if R_1 or R_2 is a field of zero characteristic and one of R_1 or R_2 is of some finite characteristic.*

Definition 2.23 *Let $R = R_1 \cup R_2$ be a biring. Then, R is said to be a bidivision ring if R is non-commutative and has no zero-divisors that is R_1 and R_2 are division rings.*

Example 5 (1) Let $R = R_1 \cup R_2$, where $R_1 = \mathcal{Z}$ and $R_2 = \mathcal{R}[x]$ the ring of integers and the ring of polynomials over $\mathcal{R}$, respectively. Since R_1 and R_2 are integral domains, it follows that R is a bidomain.

(2) The biring $R = R_1 \cup R_2$ of Example 1 is a pseudo bidomain.

(3) Let $F = F_1 \cup F_2$ where $F_1 = \mathcal{Q}(\sqrt{p_1})$, $F_2 = \mathcal{Q}(\sqrt{p_2})$ where $p_i, i = 1, 2$ are different primes. Since F_1 and F_2 are fields of zero characteristics, it follows that F is a bi-field of zero characteristic.

Theorem 2.24 *Let $R = R_1 \cup R_2$ be a biring. Then, R is a bidomain if and only if the zero bi-ideal $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal.*

Proof Suppose that R is a bidomain. Then, $R_i, i = 1, 2$ are integral domains. Since the zero ideals (0_i) in R_i are prime, it follows that $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal.

Conversely, suppose that $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal. Then, $(0_i), i = 1, 2$ are prime ideals in R_i and hence $R_i, i = 1, 2$ are integral domains. Thus $R = R_1 \cup R_2$ is a bidomain. □

Theorem 2.25 *Let $F = F_1 \cup F_2$ be a bi-field. Then, $F[x] = F_1[x] \cup F_2[x]$ is a bidomain.*

Proof Since F_1 and F_2 are fields which are integeral domains, it follows that $F_1[x]$ and $F_2[x]$ are integral domains and consequently, $F[x] = F_1[x] \cup F_2[x]$ is a bidomain. □

§3. Further Properties of Birings

Except otherwise stated in this section, all birings are assumed to be commutative with zero and unit elements.

Theorem 3.1 *Let R be any ring and let S_1 and S_2 be any two distinct subrings of R. Then, $S = S_1 \cup S_2$ is a biring.*

Proof Suppose that S_1 and S_2 are two distinct subrings of R. Then, $S_1 \nsubseteq S_2$ or $S_2 \nsubseteq S_1$ but $S_1 \cap S_2 \neq \emptyset$. Since S_1 and S_2 are rings under the same operations inherited from R, it follows that $S = S_1 \cup S_2$ is a biring. □

Corollary 3.2 *Let R_1 and R_2 be any two unrelated rings that is $R_1 \nsubseteq R_2$ or $R_2 \nsubseteq R_1$ but $R_1 \cap R_2 \neq \emptyset$. Then, $R = R_1 \cup R_2$ is a biring.*

Example 6 (1) Let $R = \mathcal{Z}$ and let $S_1 = 2\mathcal{Z}$, $S_2 = 3\mathcal{Z}$. Then, $S = S_1 \cup S_2$ is a biring.

(2) Let $R_1 = \mathcal{Z}_2$ and $R_2 = \mathcal{Z}_3$ be rings of integers modulo 2 and 3, respectively. Then, $R = R_1 \cup R_2$ is a biring.

Example 7 Let $R = R_1 \cup R_2$ be a biring, where $R_1 = \mathcal{Z}$, the ring of integers and $R_2 = C[0,1]$, the ring of all real-valued continuous functions on $[0,1]$. Let $I_1 = (p)$, where p is a prime number and let $I_2 = \{f(x) \in R_2 : f(x) = 0\}$. It is clear that I_1 and I_2 are maximal ideals of R_1 and R_2, respectively. Hence, $I = I_1 \cup I_2$ is a maximal bi-ideal of R.

Theorem 3.3 *Let $R = \{0, a, b\}$ be a set under addition and multiplication modulo 2. Then, R is a biring if and only if a and b $(a \neq b)$ are idempotent (nilpotent) in R.*

Proof Suppose that $R = \{0, a, b\}$ is a set under addition and multiplication modulo 2 and suppose that a and b are idempotent (nilpotent) in R. Let $R_1 = \{0, a\}$ and $R_2 = \{0, b\}$, where $a \neq b$. Then, R_1 and R_2 are rings and hence $R = R_1 \cup R_2$ is a biring. The proof of the converse is clear. □

Corollary 3.4 *There exists a biring of order 3.*

Theorem 3.5 *Let $R = R_1 \cup R_2$ be a finite bidomain. Then, R is a bi-field.*

Proof Suppose that $R = R_1 \cup R_2$ is a finite bidomain. Then, each $R_i, i = 1, 2$ is a finite integral domain which is a field. Therefore, R is a bifield. □

Theorem 3.6 *Let $R = R_1 \cup R_2$ be a bi-field. Then, R is a bidomain.*

Proof Suppose that $R = R_1 \cup R_2$ is a bi-field. Then, each $R_i, i = 1, 2$ is a field which is an integral domain. The required result follows from the definition of a bidomain. □

Remark 1 Every finite bidivision ring is a bi-field.

Indeed, suppose that $R = R_1 \cup R_2$ is a finite bidivision ring. Then, each $R_i, i = 1, 2$ is a finite division ring which is a field. Consequently, R is a bi-field.

Theorem 3.7 *Every biring in general need not have bi-ideals.*

Proof Suppose that $R = R_1 \cup R_2$ is a biring and suppose that $I_i, i = 1, 2$ are ideals of R_i. If $I = I_1 \cup I_2$ is such that $I_i \neq I \cap R_i$, where $i = 1, 2$, then I cannot be a bi-ideal of R. □

Corollary 3.8 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$, where $I_i, i = 1, 2$ are ideals of R_i. Then, I is a bi-ideal of R if and only if $I_i = I \cap R_i$, where $i = 1, 2$.*

Corollary 3.9 *A biring $R = R_1 \cup R_2$ may not have a maximal bi-ideal.*

Theorem 3.10 *Let $R = R_1 \cup R_2$ be a biring and let $M = M_1 \cup M_2$ be a bi-ideal of R. Then, R/M is a bi-field if and only if M is a maximal bi-ideal.*

Proof Suppose that M is a maximal bi-ideal of R. Then, each M_i, $i = 1, 2$ is a maximal ideal in R_i, $i = 1, 2$ and consequently, each R_i/I_i is a field and therefore R/M is a bi-field.

Conversely, suppose that R/M is a bi-field. Then, each R_i/M_i, $i = 1, 2$ is a field so that each M_i, $i = 1, 2$ is a maximal ideal in R_i. Hence, $M = I_1 \cup I_2$ is a maximal bi-ideal. □

Theorem 3.11 *Let $R = R_1 \cup R_2$ be a biring and let $P = P_1 \cup P_2$ be a bi-ideal of R. Then, R/P is a bidomain if and only if P is a prime bi-ideal.*

Proof Suppose that P is a prime bi-ideal of R. Then, each P_i, $i = 1, 2$ is a prime ideal in R_i, $i = 1, 2$ and so, each R_i/P_i is an integral domain and therefore R/P is a bidomain.

Conversely, suppose that R/P is a bidomain. Then, each R_i/P_i, $i = 1, 2$ is an integral domain and therefore each P_i, $i = 1, 2$ is a prime ideal in R_i. Hence, $P = P_1 \cup P_2$ is a prime bi-ideal. □

Theorem 3.12 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. If I is maximal, then I is prime.*

Proof Suppose that I is maximal. Then, $I_i, i = 1, 2$ are maximal ideals of R_i so that R_i/I_i are fields which are integral domains. Thus, $R/I = (R_1/I_1) \cup (R_2/I_2)$ is a bidomain and by Theorem 3.11, $I = I_1 \cup I_2$ is a prime bi-ideal. □

Theorem 3.13 *Let $\phi : R \to S$ be a biring homomorphism from a biring $R = R_1 \cup R_2$ onto a biring $S = S_1 \cup S_2$ and let $K = Ker\phi_1 \cup Ker\phi_2$ be the kernel of ϕ.*

(1) *If S is a bi-field, then K is a maximal bi-ideal of R;*

(2) *If S is a bidomain, then K is a prime bi-ideal of R.*

Proof By Theorem 2.7, we have $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2) \cong Im\phi = Im\phi_1 \cup Im\phi_2 = S_1 \cup S_2 = S$. The required results follow by applying Theorems 3.10 and 3.11. □

Definition 3.14 *Let $R = R_1 \cup R_2$ be a biring and let $N(R)$ be the set of nilpotent elements of R. Then, $N(R)$ is called the bi-nilradical of R if $N(R) = N(R_1) \cup N(R_2)$, where $N(R_i)$,*

$i=1,2$ are the nilradicals of R_i.

Theorem 3.15 *Let $R=R_1\cup R_2$ be a biring. Then, $N(R)$ is a bi-ideal of R.*

Proof $N(R)$ is non-empty since $0_1\in N(R_1)$ and $0_2\in N(R_2)$. Now, if $x=x_1\cup x_2, y_1\cup y_2\in N(R)$ and $r=r_1\cup r_2\in R$ where $x_i,y_i\in N(R_i), r_i\in R_i, i=1,2$, then it follows that $x-y, xr\in N(R)$. Lastly, $R_1\cap\Big(N(R_1)\cup N(R_2)\Big)=\Big(R_1\cap N(R_1)\Big)\cup\Big(R_1\cap N(R_2)\Big)=N(R_1)$. Similarly, we have $R_2\cap\Big(N(R_1)\cup N(R_2)\Big)=N(R_2)$. Hence, $N(R)$ is a bi-ideal. □

Definition 3.16 *Let $I=I_1\cup I_2$ and $J=J_1\cup J_2$ be any two bi-ideals of a biring $R=R_1\cup R_2$. The set $(I:J)$ is called a bi-ideal quotient of I and J if $(I:J)=(I_1:J_1)\cup(I_2:J_2)$, where $(I_i:J_i), i=1,2$ are ideal quotients of I_i and J_i. If $I=(0)=(0_1)\cup(0_2)$, a zero bi-ideal, then $\Big((0):J\Big)=\Big((0_1):J_1\Big)\cup\Big((0_2):J_2\Big)$ which is called a bi-annihilator of J denoted by $Ann(J)$. If $0\neq x\in R_1$ and $0\neq y\in R_2$, then $Z(R_1)=\bigcup\limits_x Ann(x)$ and $Z(R_2)=\bigcup\limits_y Ann(y)$, where $Z(R_i), i=1,2$ are the sets of zero-divisors of R_i.*

Theorem 3.17 *Let $R=R_1\cup R_2$ be a biring and let $I=I_1\cup I_2$ and $J=J_1\cup J_2$ be any two bi-ideals of R. Then, $(I:J)$ is a bi-ideal of R.*

Proof For $0=0_1\cup 0_2\in R$, we have $0_1\in(I_1:J_1)$ and $0_2\in(I_2:J_2)$ so that $(I:J)\neq\emptyset$. If $x=x_1\cup x_2, y=y_1\cup y_2\in(I:J)$ and $r=r_1\cup r_2\in R$, then $x-y, xr\in(I:J)$ since $(I_i:J_i), i=1,2$ are ideals of R_i. It can be shown that $R_1\cap\Big((I_1:J_1)\cup(I_2:J_2)\Big)=(I_1:J_1)$ and $R_2\cap\Big((I_1:J_1)\cup(I_2:J_2)\Big)=(I_2:J_2)$. Accordingly, $(I:J)$ is a bi-ideal of R. □

Example 8 Under addition and multplication modulo 6, consider the biring $R=\{0,2,3,4\}$, where $R_1=\{0,3\}$ and $R_2=\{0,2,4\}$. It is clear that $Z(R)\neq Z(R_1)\cup Z(R_2)$. Hence, for $0\neq z=x\cup y\in R$, $0\neq x\in R_1$ and $0\neq y\in R_2$, we have

$$\bigcup_{z=x\cup y} Ann(z)\neq\Big(\bigcup_x Ann(x)\Big)\cup\Big(\bigcup_y Ann(y)\Big).$$

Definition 3.18 *Let $I=I_1\cup I_2$ be any bi-ideal of a biring $R=R_1\cup R_2$. The set r(I) is called a bi-radical of I if $r(I)=r(I_1)\cup r(I_2)$, where $r(I_i), i=1,2$ are radicals of I_i. If $I=(0)=(0_1)\cup(0_2)$, then $r(I)=N(R)$.*

Theorem 3.19 *If $R=R_1\cup R_2$ is a biring and $I=I_1\cup I_2$ is a bi-ideal of R, then $r(I)$ is a bi-ideal.*

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