SMARANDACHE PSEUDO– HAPPY NUMBERS

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Happy numbers are defined by Grudman and Teeple [1], Muneer Jebral [2] and C. Asbacher [3] as:

"A natural number n is a Happy Number if the sum of squares of its digits, when added iteratively, terminates to 1." 7 is a happy number because $7^2 \rightarrow 49 \rightarrow 4^2 + 9^2 = 97 \rightarrow 9^2 + 7^2 = 130 \rightarrow 1^2 + 3^2 + 0^2 = 10 \rightarrow 1$ But 5 is not a happy number!

This paper deals with Smarandache Pseudo Happy Number, which similar to above concept, with some change in the definition. And many properties of these numbers are derived.

1.1. Definition

A natural number n is called a Smarandache Pseudo Happy Number(SPHN), if the digits of n^2, when simply added iteratively, terminates to 1; that is, the digital root of n^2 is 1

For, 8 is SPHN, because $8^2 = 64- > 6 + 4 - - > 10- > 1$ Incidentally, 7 is a happy number but it is not a SPHN !!

Now, we give a general definition of SPHN: Let $a \in N$, Let $a^2 = \sum a_i 10^i$ Let H: $N- > N$, Let $H(a) = \sum a_i$, H is a many-one function.

If $\sum a_i$, terminates to 1 when added simply and iteratively, then a is a Smarandache Pseudo Happy Number (SPHN)

1.2 The following is the set of SPHN, up to first 100 only .

Since they terminate at 1, the set of SPHN is denoted by [1].

$$[1] = \{1, 8, 10, 17, 19, 26, 28, 35, 37, 44, 46, 53, 55, 62, 64, 71, 73, 80, 82, 89, 91, 98, \ldots\ldots\}$$

We say that $H(26) = 1$ because $26 \in [1]$

Note:

(i)In general, 23 of the natural numbers are SPHN.

(ii)The negative numbers $-1, -8, -10, -17, \ldots\ldots$ are also SPHN;
But here, we will restrict to set of naturals only.

1.3. Let $[1] = a_n$
This set of SPHN is generated as: $a_1 = 1$, $a_{2n} = a_{2n-1} + 7$, $a_{2n+1} = a_{2n} + 2$, where $n \in N$

1.4.
As we notice above, 17 and 71 are both SPHN, it is obvious that the number formed by the reversal of digits of a SPHN is also a SPHN. For, the following pairs are SPHN: $(19, 91)$; $(26, 62)$; $(28, 82)$; $(35, 53)$; $(37, 73)$; $(46, 64)$; ...etc. A proof for this result is presented later on.

1.5.
Adding zeros in between or on right hand side of a SPHN do not add to the sum of the digits of the number. Hence new number, by adding zeros, is also a SPHN.
For, 17 is a SPHN. And $107^2 = 11449 \to 19 \to 1$. Hence 107 is also a SPHN.
This shows that there is infinite number of SPHN.

1.6.
Let $a_i = ith$ SPHN Then it is easy to prove the following results: (i) $a_i \equiv (mod 9)$.
(ii) $a_i^2 \equiv 1(mod 9)$.
(iii) $a_{2n-1} + a_{2n}$, when iterated, terminates to 9.
(iv) a_i, when iterated, terminates to 1 or 9
(v) $a_i \equiv 1(mod)2$.
(vi) $\| a_i$, when iterated, terminates to 1 or 8.
(vii) $a_i \bullet a_j$ is also a SPHN.
(viii) $(a_{2n})^3 + (a_{2n+1})^3$, when iterated ,terminates to 9.
(ix) $1/a_n \to 0$ as n infinity since an is an increasing sequence.

1.7.
Let $A = 1, 10, 19, 28, \ldots$ $B = 8, 17, 26, 35, \ldots$ Then $AUB = [1]$ The sequences A and B are both arithmetic progressions.

1.8.
When the digits of a SPHN are reversed, the new number is also a SPHN.
Proof. Let a be a natural number. Let $a = b_1 + b_2 \cdot 10$
$a' = b_2 + b_1 \cdot 10$
Then $a^2 = b_1^2 + 2b_1b_2 \cdot 10 + b_2^2 \cdot 100$,
And $a'^2 = b_2^2 + 2b_1b_2 \cdot 10 + b_1^2 \cdot 100$,
And the sum of the digits of

$a^2 = b_1^2 + 2b_1b_2 + b_2^2$
= sum of digits of a'^2
$= (b_1 + b_2)^2$

Hence if the number is reversed, the sum of digits remains same , and then, the new number is also SPHN.

Obviously, all the PHN palindromes are also SPHN.

Corollary *(i). Now, it is sufficient to find the square of the sum of digits of any number to test its SPHN status.*

For example, 13200432175211431501 *is a SPHN, because sum of digits of this* $20 - digit$ *number is* 46*; and* $46'^2 = 2116 \to 10 \to 1$

(ii). We have, $a^2 - a'^2 = 99 \cdot (b_1^2 - b_2^2)$ *This is another formula for finding the PHN status.*

(iii) $1, 6,$ *are triangular numbers which are SPHN;*

2.1 Non-SPHN numbers.

What about the other natural numbers which are not SPHN?

We have defined above, if the digits of n^2, when added simply and iteratively], terminates to 1. and that the set of PHN is denoted by [1]

The other numbers, when iteratively added as defined in PHN, terminate at either $4, 7$ or9. Hence the set of numbers belonging to these categories are denoted by [4], [7] or [9] respectively.

Hence we have:
$[4] = 2, 7, 11, 16, 20, 25, 29, 34 \ldots,$
$[7] = 4, 5, 13, 14, 22, 23, 31, 32 \ldots,$
$[9] = 3, 6, 9, 12, 15, 18, 21, 24 \ldots.$

2.2 We note the following:

(i) The set N of natural numbers is partitioned into [1], [4], [7] and [9]; that is, every natural number belongs to either of these sets.

(ii) No number, as added above, terminates to $2, 3, 5, 6$ or 8.

(iii) All multiples of 3 belong to [9] only.

2.3 The above sets are generated as follows: for $n \in N$,

(i) Let $[4] = b_n$, then, $b_1 = 2, b_{2n} = b_{2n-1} + 5, b_{2n+1} = b_{2n} + 4,$
(ii) Let $[7] = c_n$, then $c_1 = 3, c_{2n} = c_{2n-1} + 1, c_{2n+1} = c_{2n} + 4,$
(iii) $[9] = 3n.$

2.4 We define the multiplication [1] and [4] as:

$[1] \cdot [4] = a_r.b_r/a_r \in [1], b_r \in [4]$, i.e. the set of products of corresponding elements. The other multiplications of sets are defined similarly. Then $[1] \cdot [1] \subset [1]$, that is, $[1] \cdot [1]$. a subset of [1]

Also, $[1] \cdot [4] \subset [4],$

$[1] \cdot [7] \subset [7]$,
$[1] \cdot [9] \subset [9]$,
Considering the other products similarly, we have the following table:
[1] [4] [7] [9] ....
[1] [1] [4] [7] [9]
[4] [4] [7] [9] [9]
[7] [7] [1] [4] [9]
[9] [9] [9] [9] [9]
It is obvious from the above table, that $H^n(a) = 1$, if $a \in [1]$

2.5.

(i) Let $X = [1], [4], [7]$ Then , from the above table,$(X, \cdot)$ is an abelian group, under the subset condition, with identity as [1].

ii) Let $Y = [1], [4], [7], [9]$ Then $(Y, \cdot)$ is a monoid, under the subset condition, with identity as [1].

iii) Unfortunately, the addition of these sets, in similar way ,does not yield any definite result.

3.1 Lemma:

The sum of digits of a^3 is equal to cube of sum of digits of a. Proof: We consider a two digit number. Let $a = a_1 + a_2 \cdot 10$
$a^3 = a_1^3 + (3a_1^2 \cdot a_2) \cdot 10 + (3a_1 \cdot a_2^2) \cdot 10^2 + a_2^3.10^3$
sum of digits of
$a^3 = a_1^3 + (3a_1^2 \cdot a_2) + (3a_1 \cdot a_2^2) + a_2^3.$
$= (a_1 + a_2)^3$
$= cube of sum of digits of a.$
Hence we generalize this as: The sum of digits of a n is equal to n th power of sum of digits of a.

Now this result can be used to find the PHN status of a number As:
$(13)^6 \to (1 + 3)^6 \to 4^6 \to 4096 \to 19 \to 1.$
Therefore $(13)^6 \in [1]$, hence $(13)^6$ is a PHN
Incidentally,
$(13)^k \in [1]$, if k is a multiple of 3
$(13)^k \in [7]$, if $k = 1 + 3i, i = 1, 2, 3, \ldots$
$(13)^k \in [4]$, if $k = 2 + 3i$
Similar results can be obtained for the higher powers of any number.
Also, it can be shown that if $a^m \cdot a^n \in [i]$, then $a^{m+n} \in [i], i = 1, 4, 7, 9.$

3.2 Concatenation of SPHN.

We have, $[1] = 1, 8, 10, 17, 19, 26, 28, 35, 37, \ldots$. All the SPHN are concatenated one after another and the new number is tested.

(i) We note that:
$1 \in [1]$, $18 \in [9]$,
$1810 \in [1]$, $181017 \in [9]$,

$18101719 \in [1], 1810171926 \in [9]$,
$181017192628 \in [1], 18101719262835 \in [9]$. etc.
Hence we have, for $a_i \in [1]$,
The Concatenation $a_1 \cdot a_2 \cdot a_3 \ldots a_k \in [1]$, if k is odd, and hence it a SPHN $\in [9]$, if k is even.
(ii)A similar result is also obtained : product $a_{i+1} \cdot a_{i+2} \cdot a_{i+3} \ldots a_{i+k} \in [1]$, if$k$ is even, and it is a SPHN [9], if k is odd.

3.3 Twin Primes.
(i) The first twin primes, up to 100, are: $[5, 7], [11, 13], [17, 19].[29, 31], [41, 43], [59, 61], [71, 73]$.
The sum of each twin prime pair is a multiple of 3 Hence, The sum of each twin prime pair is a member of [9].
(ii) Let the twin primes be$2p - 1, 2p + 1, p \in N$ The product of these twin $primes = 4p^2 - 1 = 36k^2 - 1, forp = 3k$ Now the sum of digits of $36k^2 - 1$, in iteration, is 8 for all k. Hence the product belongs to [1]. Therefore the product of numbers in each twin pair is SPHN .

4.1 Change of base.
Up till now, the base of the numbers was 10.
Now change the $base \geq 2$. Then we note that the status of SPHN changes with the base. Following are some examples of numbers which are already PHN.
$35 = (55)_6 \in [1]$; Hence 35 is SPHN at the base 6 also. (additions with ref. to base 10) $71 = (107)_8 \in [1]$; Hence 71 is SPHN at the base 8.
Similarly, $89 = (118)_9 \in [1]$; Hence 89 is SPHN at the base 9.
However, some numbers, which are not SPHN with base 10, become SPHN with change of base, as: $49 = (100)_7$ is now SPHN; $50 = (62)_8$ is a SPHN

Lemma *Square of any natural number n is SPHN with ref. to n as a base.*

4.2 Product Sequences.
(i) Let S_n be a square product sequence defined as:
$S_n = 1 + s_1 \cdot s_2 \cdot s_3 \ldots s_n$, where $s_n = n^2$
we get, $S = 2, 5, 37, 577, 14401, 51849, 25401601, 1625702401 \ldots$ here, all the elements of this set, except 2 and 5, are SPHN.
(ii) Let C_n be a square product sequence defined as: $C_n = 1 + c_1 \cdot c_2 \cdot c_3 \ldots c_n$, where $c_n = n^3$
we get, $C = 2, 9, 217, 13825, 1728001, 373248001, \ldots$ here, all the elements of this set, except 2 and 9 are SPHN
(iii) Let F_n be a square product sequence defined as:
$F_n = 1 + f_1.f_2.f_3 \ldots f_n$, where $f_n = factorial \quad n$
we get, $F = 2, 3, 13, 289, 34561, 24883201, 125411328001, \ldots$
here, all the elements of this set, except 2, 3 and 13, are SPHN.

(iv) Let S be a sequence of continued sequence of natural numbers, as: $Sn = (12345\ldots n)$

That is $S = 1, 12, 123, 1234, 12345, 123456\ldots 12345\ldots n, \ldots$. If $n = 3k+1$, $k = 0, 1, 2, 3, \ldots$ then S_n is a SPHN. In all other cases, S_n belongs to [9]

(v) All factorial numbers, $(n)!$, belong to [9] because they are the multiples of 3

4.3 Summation.

We have, set $[1] = 1, 8, 10, 17, 19, 26, 28, 35, \ldots$.

This set is partitioned into two sets A and B as $A = 1, 10, 19, 28, 37, \ldots$. Its r^{th} term $a_r = 9r - 8$ and $B = 8, 17, 26, 35, 44, \ldots$, r^{th} term $b_r = 9r - 1$

now, the sum of first $2n$ terms of $A = \sum a_r = 9n(n+1)/2 - 8n$

Also, the sum of first $2n$ terms of $B = \sum b_r = 9n(n+1)/2 - n$

Hence sum of first $2n$ terms of $[1] = \sum a_r + \sum b_r = 9n^2$

Surprisingly, sum first of $2n$ terms of $[4] = 9n^2$

Also, sum of first $2n$ terms of $[7] = 9n^2$

But, sum of first $2n$ terms of $[9] = 6n^2 + 3n$.

4.4 Indices.

(i) If $a \in [1]$, then $a^k \in [1]$ for all k.

(ii) $a \in [4]$, then

$a^{3k-1} \in [7]$,

$a^{3k} \in [1]$,

$a^{3k+1} \in [4]$, for all k

(iii) $a \in [7]$, then

$a^{3k-1} \in [4]$,

$a^{3k} \in [1]$,

$a^{3k+1} \in [7]$, for all k

(iv)$a \in [9]$, $a^{3k} \in [9]$, for all k

References

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