International J.Math. Combin. Vol.1 (2010), 05-29

Radio Number of Cube of a Path

B. Sooryanarayana[1], Vishu Kumar M.[2] and Manjula K.[3]

1. Dept.of Mathematical & Computational Studies, Dr.Ambedkar Institute of Technology,
Bangalore, Karnataka State, Pin 560 056, INDIA

2. Dept.of Mathematics, Reva Institute of Technology, Yelahanka,
Bangalore, Karnataka State, Pin 560 064, INDIA

3. Dept.of Mathematics, Bangalore Institute of Technology, V. V. Puram, Bangalore,
Karnataka State,Pin 560 078, INDIA

Email: dr_bsnrao@yahoo.co.in; m_vishu78@rediffmail.com; manju_chandru2005@rediffmail.com

Abstract: Let G be a connected graph. For any two vertices u and v, let $d(u, v)$ denotes the distance between u and v in G. The maximum distance between any pair of vertices is called the diameter of G and is denoted by $diam(G)$. A Smarandachely k-radio labeling of a connected graph G is an assignment of distinct positive integers to the vertices of G, with $x \in V(G)$ labeled $f(x)$, such that $d(u, v) + |f(u) - f(v)| \geq k + diam(G)$. Particularly, if $k = 1$, such a Smarandachely radio k-labeling is called radio labeling for abbreviation. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assignment to a vertex of G. The radio number $rn(G)$ of G is minimum $\{rn(f)\}$ over all radio labelings of G. In this paper, we completely determine the radio number of the graph P_n^3 for all $n \geq 4$.

Keywords: Smarandachely radio k-labeling, radio labeling, radio number of a graph.

AMS(2010): 05C78, 05C12, 05C15

§1. Introduction

All the graphs considered here are undirected, finite, connected and simple. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d_G(u, v)$ or simply $d(u, v)$. We use the standard terminology, the terms not defined here may be found in [1].

The *eccentricity* of a vertex v of a graph G is the distance from the vertex v to a farthest vertex in G. The minimum eccentricity of a vertex in G is the *radius* of G, denoted by $r(G)$, and the of maximum eccentricity of a vertex of G is called the diameter of G, denoted by $diam(G)$. A vertex v of G whose eccentricity is equal to the radius of G is a *central vertex*.

For any real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x and $\lfloor x \rfloor$

[1]Received January 5, 2010. Accepted March 20, 2010.

denotes the greatest integer less than or equal to x. We recall that k^{th} power of a graph G, denoted by G^k is the graph on the vertices of G with two vertices u and v are adjacent in G^k whenever $d(u,v) \leq k$. The graph G^3 is called a cube of G.

A labeling of a connected graph is an injection $f : V(G) \to Z^+$, while a Smarandache k-radio labeling of a connected graph G is an assignment of distinct positive integers to the vertices of G, with $x \in V(G)$ labeled $f(x)$, such that $d(u,v)+|f(u)-f(v)| \geq k+diam(G)$. Particularly, if $k = 1$, such a Smarandache radio k-labeling is called radio labeling for abbreviation. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assigned to a vertex of G. The *radio number* $rn(G)$ of G is $min\{rn(f)\}$, over all radio labelings f of G. A radio labeling f of G is a *minimal radio labeling* of G if $rn(f) = rn(G)$.

Radio labeling is motivated by the channel assignment problem introduced by Hale et al [10] in 1980. The radio labeling of a graph is most useful in FM radio channel restrictions to overcome from the effect of noise. This problem turns out to find the minimum of maximum frequencies of all the radio stations considered under the network.

The notion of radio labeling was introduced in 2001, by G. Chartrand, David Erwin, Ping Zhang and F. Harary in [2]. In [2] authors showed that if G is a connected graph of order n and diameter two, then $n \leq rn(G) \leq 2n-2$ and that for every pair k, n of integers with $n \leq k \leq 2n-2$, there exists a connected graph of order n and diameter two with $rn(G) = k$. Also, in the same paper a characterization of connected graphs of order n and diameter two with prescribed radio number is presented.

In 2002, Ping Zhang [15] discussed upper and lower bounds for a radio number of cycles. The bounds are showed to be tight for certain cycles. In 2004, Liu and Xie [5] investigated the radio number of square of cycles. In 2007, B. Sooryanarayana and Raghunath P [12] have determined radio labeling of cube of a cycle, for all $n \leq 20$, all even $n \geq 20$ and gave bounds for other cycles. In [13], they also determined radio number of the graph C_n^4, for all even n and odd $n \leq 25$.

A radio labeling is called *radio graceful* if $rn(G) = n$. In [12] and [13] it is shown that the graph C_n^3 is radio graceful if and only if $n \in \{3,4,5,6,7,9,10,11,12,13,18,19\}$ and C_n^4 is radio graceful if and only if $n \in \{3,4,5,6,7,8,9,11,12,13,14,15,16,17,23,24,25\}$.

In 2005, D. D. F. Liu and X. Zhu [7] completely determined radio numbers of paths and cycles. In 2006, D. D. F. Liu [8] obtained lower bounds for the radio number of trees, and characterized the trees achieving this bound. Moreover in the same paper, he gave another lower bound for the radio number of the trees with at most one vertex of degree more than two (called spiders) in terms of the lengths of their legs and also characterized the spiders achieving this bound.

The results of D. D. F. Liu [8] generalizes the radio number for paths obtained by D. D. F. Liu and X. Zhu in [7]. Further, D.D.F. Liu and M. Xie obtained radio labeling of square of paths in [6]. In this paper, we completely determine the radio labeling of cube of a path. The main result we prove in this paper is the following Theorem 1.1. The lower bound is established in section 2 and a labeling procedure is given in section 3 to show that the lower bounds achieved in section 3 are really the tight upper bounds.

Theorem 1.1 *Let P_n^3 be a cube of a path on n ($n \geq 6$ and $n \neq 7$) vertices. Then*

$$rn(P_n^3)=\begin{cases} \frac{n^2+12}{6}, & if\ n \equiv 0\ (mod\ 6) \\ \frac{n^2-2n+19}{6}, & if\ n \equiv 1\ (mod\ 6) \\ \frac{n^2+2n+10}{6}, & if\ n \equiv 2\ (mod\ 6) \\ \frac{n^2+15}{6}, & if\ n \equiv 3\ (mod\ 6) \\ \frac{n^2-2n+16}{6}, & if\ n \equiv 4\ (mod\ 6) \\ \frac{n^2+2n+13}{6}, & if\ n \equiv 5\ (mod\ 6) \end{cases}.$$

We recall the following results for immediate reference.

Theorem 1.2(Daphne Der-Fen Liu, Xuding Zhu [6]) *For any integer $n \geq 4$,*

$$rn(P_n) = \begin{cases} 2k^2+3, & if\ \ n = 2k+1; \\ 2k^2-2k+2 & if\ n = 2k \end{cases}$$

Lemma 1.3(Daphne Der-Fen Liu, Melanie Xie [7]) *Let P_n^2 be a square path on n vertices with $k = \lfloor \frac{n}{2} \rfloor$. Let $\{x_1, x_2, \ldots, x_n\}$ be a permutation of $V(P_n^2)$ such that for any $1 \leq i \leq n-2$,*

$$min\{d_{P_n}(x_i, x_{i+1}),\ d_{P_n}(x_{i+1}, x_{i+2}) \leq k+1,$$

and if k is even and the equality in the above holds, then $d_{P_n}(x_i, x_{i+1})$ and $d_{P_n}(x_{i+1}, x_{i+2})$ have different parities. Let f be a function, $f: V(P_n^2) \to \{0, 1, 2, \ldots\}$ with $f(x_1) = 0$, and $f(x_{i+1}) - f(x_i) = k+1-d(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. Then f is a radio-labeling for P_n^2.

§2. Lower Bound

In this section we establish the lower bound for Theorem 1.1. Throughout, we denote a path on n vertices by P_n, where $V(P_n) = \{v_1, v_2, v_3\ .\ .\ , v_n\}$ and $E(P_n)$ =$\{v_iv_{i+1} \mid i = 1, 2, ..., n-1\}$. A path on odd length is called an *odd path* and that of even length is called an *even path.*

Observation 2.1 By the definition of P_n^3, for any two vertices u, $v \in V(P_n^3)$, we get

$$d_{P_n^3}(u, v) = \left\lceil \frac{d_{P_n}(u,v)}{3} \right\rceil \text{ and } diam(P_n^3) = \left\lceil \frac{n-1}{3} \right\rceil$$

Observation 2.2 An odd path P_{2k+1} on $2k+1$ vertices has exactly one center namely v_{k+1}, while an even path P_{2k} on $2k$ vertices has two centers v_k and v_{k+1}.

For each vertex $u \in V(P_n^3)$, the level of u, denoted by $l(u)$, is the smallest distance in P_n from u to a center of P_n. Denote the level of the vertices in a set A by $L(A)$.

Observation 2.3 For an even n, the distance between two vertices v_i and v_j in P_n^3 is given by their corresponding levels as;

$$d(v_i, v_j) = \begin{cases} \left\lceil \frac{|l(v_i)-l(v_j)|}{3} \right\rceil, & \text{whenever } 1 \le i,j \le \frac{n}{2} \text{ or } \frac{n+2}{2} \le i,j \le n \\ \\ \left\lceil \frac{l(v_i)+l(v_j)+1}{3} \right\rceil, & \text{otherwise} \end{cases} \tag{1}$$

Observation 2.4 For an odd n, the distance between two vertices v_i and v_j in P_n^3 is given by their corresponding levels as;

$$d(v_i, v_j) = \begin{cases} \left\lceil \frac{|l(v_i)-l(v_j)|)}{3} \right\rceil, & \text{whenever } 1 \le i,j \le \frac{n+1}{2} \text{ or } \frac{n+1}{2} \le i,j \le n \\ \\ \left\lceil \frac{l(v_i)+l(v_j)}{3} \right\rceil, & \text{otherwise} \end{cases} \tag{2}$$

Observation 2.5 If n is even, then

$$L(V(P_n^3)) = \left\{\tfrac{n}{2}-1, \tfrac{n}{2}-2, \ldots, 2, 1, 0, 0, 1, 2, \ldots, \tfrac{n}{2}-2, \tfrac{n}{2}-1\right\}.$$

Therefore

$$\sum_{v_i \in V(P_n^3)} l(v_i) = 2\left[1+2+\cdots+\frac{n}{2}-1\right] = \frac{n}{2}\left\{\frac{n}{2}-1\right\} = \frac{n^2-2n}{4} \tag{3}$$

Observation 2.6 If n is odd, then

$$L(V(P_n^3) = \left\{\tfrac{n-1}{2}, \tfrac{n-1}{2}-1, \ldots 2, 1, 0, 1, 2, \ldots \tfrac{n-1}{2}-1, \tfrac{n-1}{2}\right\}.$$

Therefore

$$\sum_{v_i \in V(P_n^3)} l(v_i) = 2\left[1+2+\cdots+\frac{n-1}{2}\right] = \frac{n-1}{2}\left\{\frac{n+1}{2}\right\} = \frac{n^2-1}{4} \tag{4}$$

Let f be a radio labeling of the graph P_n^3. Let $x_1, x_2, \ldots, x_n$ be the sequence of the vertices of P_n^3 such that $f(x_{i+1}) > f(x_i)$ for every $i, 1 \le i \le n-1$. Then we have

$$f(x_{i+1}) - f(x_i) \ge diam(P_n^3) + 1 - d(x_{i+1}, x_i) \tag{5}$$

for every $i, 1 \le i \le n-1$.

Summing up $n-1$ inequalities in (5), we get

$$\sum_{i=1}^{n-1}[f(x_{i+1}) - f(x_i)] \ge \sum_{i=1}^{n-1}[diam(P_n^3)+1] - \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \tag{6}$$

The terms in the left hand side of the inequality (6) cancels each other except the first and the last term, therefore, inequality (6) simplifies to

$$f(x_n) - f(x_1) \ge (n-1)[diam(P_n^3)+1] - \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \tag{7}$$

If f is a minimal radio labeling of P_n^3, then $f(x_1) = 1$ (else we can reduce the span of f by $f(x_n) - f(x_1) + 1$ by reducing each label by $f(x_1) - 1$). Therefore, inequality (7) can be written as

$$f(x_n) \geq (n-1)[diam(P_n^3) + 1] - \sum_{i=1}^{n-1} d(x_{i+1}, x_i) + 1 \tag{8}$$

By the observations 2.3 and 2.4, for every i, $1 \leq i \leq n-1$ it follows that

$$d(x_{i+1}, x_i) \leq \left\lceil \frac{l(x_{i+1}) + l(x_i) + 1}{3} \right\rceil \leq \frac{l(x_{i+1}) + l(x_i)}{3} + 1 \tag{9}$$

whenever n is even. And

$$d(x_{i+1}, x_i) \leq \left\lceil \frac{l(x_{i+1}) + l(x_i)}{3} \right\rceil \leq \frac{l(x_{i+1}) + l(x_i)}{3} + \frac{2}{3} \tag{10}$$

whenever n is odd.

Inequalities (9) and (10), together gives,

$$\sum_{i=1}^{n-1} d(x_{i+1}, x_i) \leq \sum_{i=1}^{n-1} \left[\frac{l(x_{i+1}) + l(x_i)}{3} + k \right]$$

where $k = 1$, if n is even and $k = \frac{2}{3}$ if n is odd.

$$\Rightarrow \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \quad \leq \quad \frac{1}{3} \times 2 \sum_{i=1}^{n} l(x_i) - \frac{1}{3}\left[l(x_n) + l(x_1)\right] + k(n-1)$$

$$\Rightarrow \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \quad \leq \quad \frac{2}{3} \sum_{i=1}^{n} l(x_i) + k(n-1) - \frac{1}{3}\left[l(x_1) + l(x_n)\right] \tag{11}$$

From the inequalities 8 and 11, we get

$$f(x_n) \geq (n-1)[diam(P_n^3) + 1] - \frac{2}{3} \sum_{i=1}^{n} l(x_i) + 1 - k(n-1) + \frac{1}{3}\left[l(x_1) + l(x_n)\right]$$

$$\Rightarrow f(x_n) \geq (n-1) diam(P_n^3) - \frac{2}{3} \sum_{i=1}^{n} l(x_i) + 1 + (1-k)(n-1) + \frac{1}{3}\left[l(x_1) + l(x_n)\right] \tag{12}$$

We now observe that the equality between the second and third terms in (9) holds only if $l(x_{i+1}) + l(x_i) \equiv 0 \pmod 3$ and the equality between the second and third terms in holds only if(10) $l(x_{i+1}) + l(x_i) \equiv 1 \pmod 3$. Therefore, there are certain number of pairs (x_{i+1}, x_i) for which the strict inequality holds. That is, the right hand side of (9) as well as (10) will exceed by certain amount say ξ. Thus, the right hand side of (12) can be refined by adding an amount ξ as;

$$f(x_n) \geq \left\lceil (n-1) diam(P_n^3) - \frac{2}{3} \sum_{i=1}^{n} l(x_i) + 1 + (1-k)(n-1) + \eta + \xi \right\rceil \tag{13}$$

where $\eta = \frac{1}{3}\left[l(x_i) + l(x_{i+1})\right]$.

We also see that the value of ξ increases heavily if we take a pair of vertices on same side of the central vertex. So, here onwards we consider only those pairs of vertices on different sides of a central vertex.

Observation 2.7 All the terms in the right side of the inequality (13), except ξ and η, are the constants for a given path P_n. Therefore, for a tight lower bound these quantities must be minimized. If n is even, we have two cental vertices and hence a minimal radio labeling will start the label from one of the central vertices and end at the other vertex, so that $l(x_1) = l(x_n) = 0$. However, if n is odd, as the graph P_n has only one central vertex, either $l(x_1) > 0$ or $l(x_n) > 0$. Thus, $\eta \geq 0$ for all even n, and $\eta \geq \frac{1}{3}$ for all odd n.

The terms η and ξ included in the inequality (13) are not independent. The choice of initial and final vertices for a radio labeling decides the value of η, but at the same time it (this choice) also effect ξ (since ξ depends on the levels in the chosen sequence of vertices). Thus, for a minimum span of a radio labeling, the sum $\eta + \xi$ to be minimized rather than η or ξ.

Observation 2.8 For each j, $0 \leq j \leq 2$, define $L_j = \{v \in V(P_n^3) | l(v) \equiv j (mod\ 3)\}$ and for each pair $(x_{i+1}, x_i), 1 \leq i \leq n-1$ of vertices of $V(P_n^3)$, let

$$\xi_i = \left\{\frac{l(x_{i+1})+l(x_i)}{3} + 1\right\} - \left\lceil \frac{l(x_{i+1})+l(x_i)+1}{3} \right\rceil, \text{ if n is even, or}$$
$$\xi_i = \left\{\frac{l(x_{i+1})+l(x_i)}{3} + \frac{2}{3}\right\} - \left\lceil \frac{l(x_{i+1})+l(x_i)}{3} \right\rceil, \text{ if n is odd.}$$

Then there are following three possible cases:

Possibility 1: Either (i) both $x_{i+1}, x_i \in L_0$ or (ii) one of them is in L_1 and the other is in L_2. In this case

$$\xi_i = \begin{cases} 0, & if\ n\ is\ even \\ \frac{2}{3}, & if\ n\ is\ odd \end{cases}.$$

Possibility 2: Either (i) both $x_{i+1}, x_i \in L_2$ or (ii) one of them is in L_0 and the other is in L_1. In this case

$$\xi_i = \begin{cases} \frac{1}{3}, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}.$$

Possibility 3: Either (i) both $x_{i+1}, x_i \in L_1$ or (ii) one of them is in L_0 and the other is in L_2. In this case

$$\xi_i = \begin{cases} \frac{2}{3}, & if\ n\ is\ even \\ \frac{1}{3}, & if\ n\ is\ odd \end{cases}.$$

Observation 2.9 For the case n is even, the Possibility 1 given in the Observation 2.8 holds for every pair of consecutive vertices in the sequence of the form either $l_{\alpha_1}, r_{\alpha_2}, l_{\alpha_3}, r_{\alpha_4}, l_{\alpha_5}, r_{\alpha_6}, \ldots,$ or $l_{\beta_1}, r_{\gamma_1}, l_{\beta_2}, r_{\gamma_2}, l_{\beta_3}, r_{\gamma_3}, \ldots,$, where l_{α_i}, l_{β_i}, l_{γ_i} denote the vertices in the left of a central vertex and at a level congruent to 0, 1, 2 under modulo 3 respectively, and, r_{α_i},r_{β_i}, r_{γ_i} denote the corresponding vertices in the right side of a central vertex of the path P_n. The first sequence covers only those vertices of P_n^3 which are at a level congruent to 0 under modulo 3, and, the second sequence covers only those vertices of L_1 (or L_2) which lie entirely on one side of a central vertex. Now, as the sequence $x_1, x_2, \ldots, x_n$ covers the entire vertex set of P_n^3, the sequence should have at least one pair as in Possibility 2 (taken this case for minimum ξ_i) to link a vertex in level congruent to 0 under modulo 3 with a vertex not at a level congruent to 0 under modulo 3. For this pair $\xi_i \geq \frac{1}{3}$. Further, to cover all the left as well as right vertices in the same level congruent to $i, 1 \leq i \leq 2$, we again require at least one pairs as in Possibility 2 or 3. Thus, for this pair again we have $\xi_i \geq \frac{1}{3}$. Therefore,

$$\xi = \sum_{i=1}^{n} \xi_i \geq \frac{2}{3}$$

for all even n.

The above Observation 2.9 can be visualize in the graph called *level diagram* shown in Figures 1 and 2. A Hamilton path shown in the diagram indicates a sequence $x_1, x_2, \ldots, x_n$ where thin edges join the pair of vertices as in Possibility 1 indicted in Observation 2.8 and the bold edges are that of Possibility 2 or 3. Each of the subgraphs $G_{0,0}$, $G_{1,2}$ and $G_{2,1}$ is a complete bipartite graph having only thin edges and $s = \lceil \frac{n-4}{6} \rceil$.

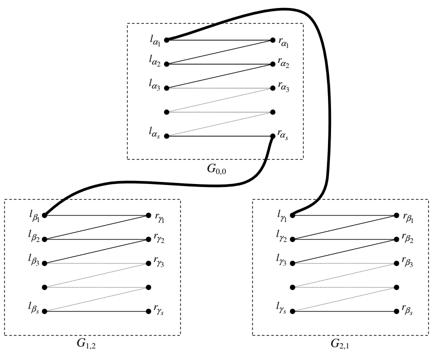


Figure 1: For P_n^3 when $n \equiv 0$ or 2 (mod 6).

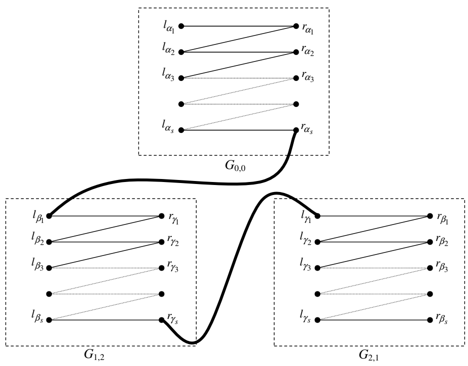


Figure 2: For P_n^3 when $n \equiv 0$ or 2 (mod 6).

If $\xi = \frac{2}{3}$ and $n \equiv 0$ (mod 6), then only two types of Hamilton paths are possible as shown in Figures **??** and **??**. In each of the case either $l(x_1) > 0$ or $l(x_n) > 0$, therefore $\eta \geq \frac{1}{3}$. Hence $\eta + \xi \geq 1$ in this case.

If $\eta = 0$, then both the starting and the ending vertices should be in the subgraph $G_{0,0}$. Thus, one of the thin edges in $G_{0,0}$ to be broken and one of its ends to be joined to a vertex in $G_{1,2}$ and the other to a vertex in $G_{2,1}$ with bold edges. These two edges alone will not connect the subgraphs, so to connect $G_{1,2}$ and $G_{2,1}$ we need at least one more bold edge. Therefore, $\xi \geq 1$ and hence $\eta + \xi \geq 1$ in this case also.

In all the other possibilities for the case $n \equiv 0$ or 2 under modulo 6, we have $\eta \geq \frac{1}{3}$ and $\xi \geq \frac{2}{3}$, so clearly $\eta + \xi \geq 1$.

The situation is slightly different for the case when $n \equiv 4 \pmod 6$. In this case;

If $\xi = \frac{2}{3}$, then there is one and only one possible type of Hamilton path as shown in Figure **??**, so $l(x_1) > 0$ and $l(x_n) > 0$ implies that $\eta \geq \frac{2}{3}$ and hence $\eta + \xi \geq \frac{4}{3}$.

Else if, $\eta = 0$, then two bold edges are required. One edge is between a vertex of $G_{1,2}$ and a vertex of $G_{0,0}$, and, the other edge between a vertex of $G_{2,1}$ and a vertex of $G_{0,0}$ (for each such edges $\xi_i \geq \frac{1}{3}$). These two edges will not connect all the subgraphs. For this, we require an edge between a vertex of $G_{1,2}$ and a vertex of $G_{2,1}$, which can be done minimally only by an edge between a pair of vertices as in Possibility 3 indicated in observation 2.8 (for such an edge $\xi_i = \frac{2}{3}$). Thus, $\xi \geq 2 \times \frac{1}{3} + \frac{2}{3} = \frac{4}{3}$.

If $\xi = 1$, then the possible Hamilton path should contain at least either (i) one edge between $G_{1,2}$ and $G_{2,1}$, and, another edge from $G_{0,0}$, or, (ii) three edges from $G_{0,0}$. The first case is impossible because we can not join the vertices that lie on the same side of a central vertex with $\xi = 1$ and the second case is possible only if $\eta \geq \frac{1}{3}$.

Hence, for all even n, we get

$$\eta + \xi \geq 1, \quad if \quad n \equiv 0 \ or \ 2 \ (mod \ 6) \tag{14}$$

$$\eta + \xi \geq \frac{4}{3}, \quad if \quad n \equiv 4 \ (mod \ 6) \tag{15}$$

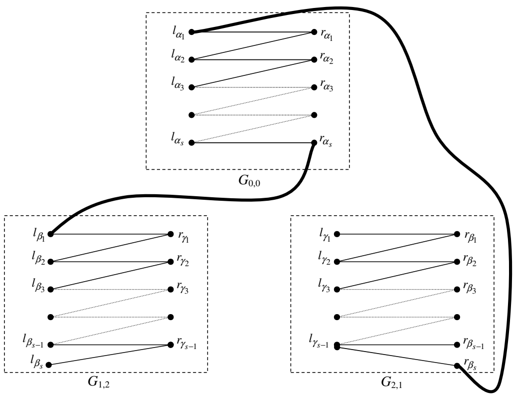


Figure 3: A Hamilton path in a level graph for the case $n \equiv 4 \pmod 6$

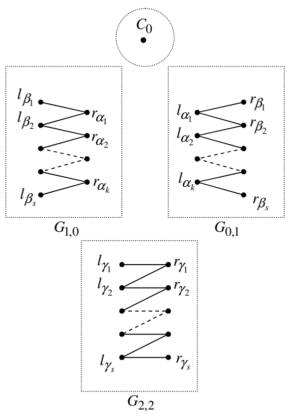


Figure 4: Level graph for the case $n \equiv 3$ or 5 (mod 6).

Observation 2.10 For the case n is odd, the Possibility 2 given in observation 2.8 holds for every pair of consecutive vertices in the sequence of the form l_{β_1}, r_{α_1}, l_{β_2}, r_{α_2}, l_{β_3}, r_{α_3}, $\ldots$, or r_{β_1}, l_{α_1}, r_{β_2}, l_{α_2}, r_{β_3}, l_{α_3}, $\ldots$, or l_{γ_1}, r_{γ_1} and l_{γ_2}, r_{γ_2}, l_{γ_3}, r_{γ_3}, $\ldots$, where l_{α_i}, l_{β_i}, l_{γ_i} denote the vertices in the left of a central vertex and at a level congruent to 0, 1, 2 under modulo 3 respectively, and, r_{α_i},r_{β_i} and r_{γ_i} denote the corresponding vertices in the right side of a central vertex of the path P_n. Let C_0 be the central vertex. Then C_0 can be joined to one of the first two sequences or the first sequence can be combined with second sequence through C_0. The third sequence covers only those vertices of P_n^3 which are at a level congruent to 2 under modulo

3, and, the first two sequences are disjoint. Hence to get a Hamilton path $x_1, x_2, \ldots, x_n$ to cover the entire vertex set of P_n^3, it should have at least a pair as in Possibility 3, (if the vertex C_0 combines first and second sequences) or at least two pairs that are not as in Possibility 1. Therefore, as the graph contains only one center vertex,

$$\eta \geq \frac{1}{3} \text{ and } \xi \geq \frac{1}{3}$$

The above observation 2.10 will be visualized in the Figure 4.

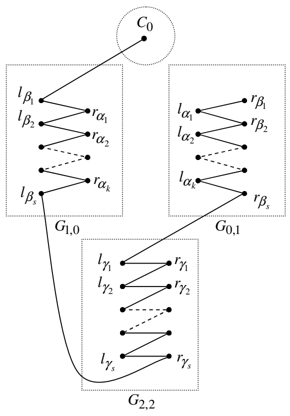
In either of the cases, we claim that $\eta + \xi \geq \frac{5}{3}$

We note here that, if we take more than three edges amongst $G_{1,0}$, $G_{0,1}$ and $G_{2,2}$ in the level graphs shown in Figure 4, then $\xi \geq 4 \times \frac{1}{3}$, so the claim follows immediately as $\eta \geq \frac{1}{3}$.

Case 1: If $\eta = \frac{1}{3}$, then $l(x_1) = 0$, so the vertex C_0 is in either first sequence or in the second sequence (as mentioned in the Observation 2.10), but not in both. Hence at least two edges are required to get a Hamilton path. The minimum possible edges amongst $G_{1,0}$, $G_{0,1}$ and $G_{2,2}$) are discussed in the following cases.

Subcase 1.1: With two edges

The only possible two edges (in the sense of minimum ξ) are shown in Figure 5. Thus, $\xi \geq \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$. Hence the claim.



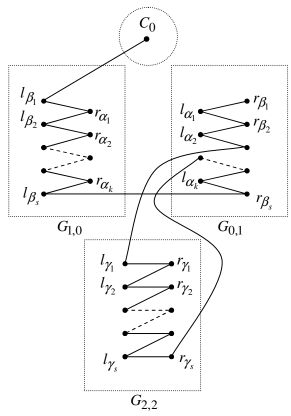


Figure 5: Level graph($n \equiv 3$ or 5 (mod 6)). Figure 6: hamilton cycle($\eta = \frac{1}{3}$, $n \equiv 3,\ 5 \pmod 6$).

Subcase 1.2: With three edges

The only possible three edges are shown in Figures 5 and 6. In each case, $\xi \geq \frac{4}{3}$. Hence the claim.

Case 2: If $\eta = \frac{2}{3}$, then either $l(x_1) = 0$ and $l(x_n) = 2$, or, $l(x_1) = 1$ and $l(x_n) = 1$. In the first case at least two edges are necessary, both these edges can not be as in Possibility 3 (because two such edges disconnect $G_{0,1}$ or disconnect $G_{2,2}$ or form a tree with at least three end vertices as shown in Figure 7. Similar fact holds true for the second case also (Follows easily from Figure 8.

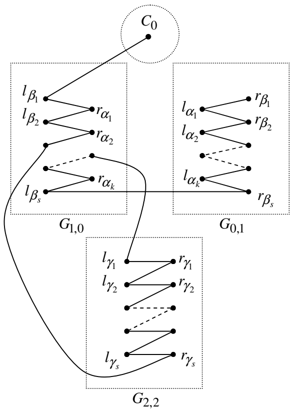


Figure 7: hamilton cycle for the case $\eta = \frac{1}{3}$ and $n \equiv 3$ or $5 (\mathrm{mod}\ 6)$.

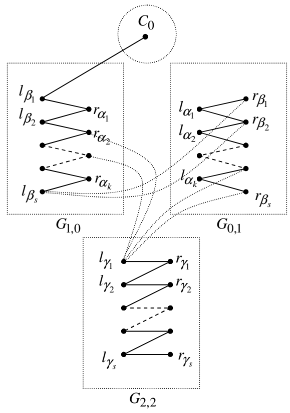


Figure 8: hamilton cycle for the case $\eta = \frac{2}{3}$ and $n \equiv 3$ or $5 (\mathrm{mod}\ 6)$.

Hence $\xi \geq \frac{1}{3} + \frac{2}{3}$. Therefore,

$$\xi + \eta \geq \frac{5}{3} \quad for \quad n \equiv 3 \ or \ 5 \ (mod \ 3) \tag{16}$$

The case $n \equiv 1 (mod\ 3)$ follows similarly.

We now prove the necessary part of the Theorem 1.1.

Case 1: $n \equiv 0$ (mod 6) and $n \geq 6$

Substituting the minimum possible bound for $\eta + \xi = 1$ (as in equation (14)) $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-2n}{4}$ (follows by Observation 2.5) and $k = 1$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n}{3} - \frac{2}{3}\left(\frac{n^2-2n}{4}\right) + 1 + 1 \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{n^2}{6} + 2 \right\rceil = \frac{n^2}{6} + 2 \tag{17}$$

Hence $rn(P_n^3) \geq \frac{n^2+12}{6}$, whenever $n \equiv 0$ (mod 6) and $n \geq 6$.

Case 2: $n \equiv 1$ (mod 6) and $n \geq 13$

Substituting the minimum possible bound for $\eta + \xi = \frac{5}{3}$ (as in equation (14)), $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n-1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-1}{4}$ and $k = \frac{2}{3}$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\left(\frac{n-1}{3}\right) - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + 1 + \frac{1}{3}(n-1) + \frac{5}{3} \right\rceil$$

$$\Rightarrow f(x_n) \geq (n-1)\left(\frac{n-1}{3}\right) - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + \frac{1}{3}(n-1) + 3 = \frac{n^2-2n+19}{6} \tag{18}$$

Hence $rn(P_n^3) \geq \frac{n^2-2n+19}{6}$, whenever $n \equiv 1 \pmod 6$ and $n \geq 13$.

Case 3: $n \equiv 2 \pmod 6$ and $n \geq 8$

Substituting the minimum possible bound for $\eta+\xi = 1$ (as in equation (14)), $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n+1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-2n}{4}$ and $k = 1$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n+1}{3} - \frac{2}{3}\left(\frac{n^2-2n}{4}\right) + 1 + 1 \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{(n-2)^2}{6} + n + 1 \right\rceil = \frac{(n-2)^2}{6} + n + 1 = \frac{n^2+2n+10}{6} \tag{19}$$

Hence $rn(P_n^3) \geq \frac{n^2+2n+10}{6}$, whenever $n \equiv 2 \pmod 6$ and $n \geq 8$.

Case 4: $n \equiv 3 \pmod 6$ and $n \geq 9$

Substituting $\eta + \xi = \frac{5}{3}$, $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-1}{4}$ and $k = \frac{2}{3}$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n}{3} - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + 1 + \frac{1}{3}(n-1) + \frac{5}{3} \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{(n-3)^2}{6} + n + 1 \right\rceil = \frac{(n-3)^2}{6} + n + 1 = \frac{n^2+15}{6} \tag{20}$$

Hence $rn(P_n^3) \geq \frac{n^2+15}{6}$, whenever $n \equiv 3 \pmod 6$ and $n \geq 9$.

Case 5: $n \equiv 4 \pmod 6$ and $n \geq 10$

Substituting the minimum possible bound for $\eta+\xi = \frac{4}{3}$ (as in equation (15)), $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n-1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-2n}{4}$ and $k = 1$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n-1}{3} - \frac{2}{3}\left(\frac{n^2-2n}{4}\right) + 1 + \frac{4}{3} \right\rceil$$

$$f(x_n) \geq \left\lceil \frac{(n-4)^2}{6} + n \right\rceil = \frac{(n-4)^2}{6} + n = \frac{n^2-2n+16}{6} \tag{21}$$

Hence $rn(P_n^3) \geq \frac{n^2-2n+16}{6}$, whenever $n \equiv 4 \pmod 6$ and $n \geq 10$.

Case 6: $n \equiv 5 \pmod 6$ and $n \geq 11$

Substituting $\eta + \xi = \frac{5}{3}$, $diam(G) = \lceil \frac{n-1}{3} \rceil = \frac{n+1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-1}{4}$ and $k = \frac{2}{3}$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n+1}{3} - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + 1 + \frac{1}{3}(n-1) + \frac{5}{3} \right\rceil = \left\lceil \frac{n^2+2n+13}{6} \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{(n-5)^2}{6} + 2(n-1) \right\rceil = \frac{(n-5)^2}{6} + 2(n-1) = \frac{n^2+2n+13}{6} \tag{22}$$

Hence $rn(P_n^3) \geq \frac{n^2+n+13}{6}$, whenever $n \equiv 5 \pmod 6$ and $n \geq 11$.

§3. Upper Bound and Optimal Radio-Labelings

We now establish Theorem 1.1, it suffices to give radio-labelings that achieves the desired spans. Further, we will prove the following lemma similar to the Lemma 1.3 of Daphne Der-Fen Liu and Melanie Xie obtained in [7].

Lemma 3.1 *Let P_n^3 be a cube path on n $(n \geq 6)$ vertices with $k = \lceil \frac{n-1}{3} \rceil$. Let $\{x_1, x_2, \ldots, x_n\}$ be a permutation of $V(P_n^3)$ such that for any $1 \leq i \leq n-2$,*

$$min\{d_{P_n}(x_i, x_{i+1}),\ d_{P_n}(x_{i+1}, x_{i+2})\} \leq 3\frac{k}{2} + 1$$

if k is even and the equality in the above holds, then the sum of the parity congruent to 0 under modulo 3. Let f be a function, $f : V(P_n^3) \to \{1, 2, 3, \ldots\}$ with $f(x_1) = 1$, and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$, where $d(x_i, x_{i+1}) = d_{P_n^3}(x_i, x_{i+1})$. Then f is a radio-labeling for P_n^3.

Proof Recall, $diam(P_n^3) = k$. Let f be a function satisfying the assumption. It suffices to prove that $f(x_j) - f(x_i) \geq k + 1 - d(x_i, x_j)$ for any $j \geq i + 2$. For $i = 1, 2, \ldots, n-1$, set

$$f_i = f(x_{i+1}) - f(x_i).$$

Since the difference in two consecutive labeling is at least one it follows that $f_i \geq 1$. Further, for any $j \geq i + 2$, it follows that

$$f(x_j) - f(x_i) = f_i + f_{i+1} + \cdots + f_{j-1}.$$

Suppose $j = i + 2$. Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$. (The proof for $d(x_{i+1}, x_{i+2}) \geq d(x_i, x_{i+1})$ is similar.) Then, $d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$. Let $x_i = v_a$, $x_{i+1} = v_b$, and $x_{i+2} = v_c$. It suffices to consider the following cases.

Case 1: $b < a < c$ or $c < a < b$

Since $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$, we obtain $d(x_i, x_{i+1}) = d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$ and $d_{P_n}(x_i, x_{i+2}) \leq 2$ so, $d(x_i, x_{i+2}) = 1$. Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &\geq 2k + 2 - 2\left(\frac{k+2}{2}\right) \\ &= k + 1 - 1 \\ &= k + 1 - d(x_i, x_{i+2}) \end{aligned}$$

Case 2: $a < b < c$ or $c < b < a$

In this case, $d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) - 1$ and hence

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &= 2k + 2 - \{d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})\} \\ &= 2k + 2 - \{d(x_i, x_{i+2}) + 1\} \\ &= 2k + 1 - d(x_i, x_{i+2}) \\ &\geq k + 1 - d(x_i, x_{i+2}) \end{aligned}$$

Case 3: $a < c < b$ or $b < c < a$

Assume $min\{d_{P_n}(x_i, x_{i+1}),\ d_{P_n}(x_{i+1}, x_{i+2})\} < 3\frac{k}{2} + 1$, then we have $d(x_{i+1}, x_{i+2}) < \frac{k+2}{2}$ and by triangular inequality,

$$d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2})$$

Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &= 2k + 2 - [d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2})] - 2d(x_{i+1}, x_{i+2}) \\ &\geq 2k + 2 - [d(x_i, x_{i+2})] - 2d(x_{i+1}, x_{i+2}) \\ &> 2k + 2 - d(x_i, x_{i+2}) - 2\left(\frac{k+2}{2}\right) \\ &= k - d(x_i, x_{i+2}) \end{aligned}$$

Therefore,

$$f(x_{i+2}) - f(x_i) \geq k + 1 - d(x_i, x_{i+2})$$

If $min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} = 3\frac{k}{2} + 1$, then by our assumption, it must be that $d_{P_n}(x_{i+1}, x_{i+2}) = 3\frac{k}{2} + 1$ (so k is even) ,and, sum of $d_{P_n}(x_i, x_{i+1})$ and $d_{P_n}(x_{i+1}, x_{i+2})$ is congruent to 0 under modulo 3 implies that $d_{P_n}(x_i, x_{i+1}) \not\equiv 0 \pmod 3$. Hence, we have

$$d(x_i, x_{i+2}) = d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) + 1.$$

This implies

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= 2(k+1) - [d(x_i, x_{i+2})] - d(x_{i+1}, x_{i+2}) - d(x_{i+1}, x_{i+2}) + 1 \\ &\geq 2k + 2 - 2[d(x_{i+1}, x_{i+2})] - d(x_i, x_{i+2}) + 1 \\ &\geq 2k + 2 - 2\left(\frac{k+2}{2}\right) - d(x_i, x_{i+2}) + 1 \\ &= k + 1 - d(x_i, x_{i+2}) \end{aligned}$$

Let $j = i + 3$. First, we assume that the sum of some pairs of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, $d(x_{i+2}, x_{i+3})$ is at most $k + 2$. Then

$$d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_{i+2}, x_{i+3}) \leq (k+2) + k = 2k + 2$$

and hence,

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k + 3 - d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) - d(x_{i+2}, x_{i+3}) \\ &\geq 3k + 3 - (2k+2) \\ &= k + 1 > k + 1 - d(x_i, x_{i+3}). \end{aligned}$$

Next, we assume that the sum of every pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$ and $d(x_{i+2}, x_{i+3})$ is greater than $k+2$. Then, by our hypotheses, it follows that

$$d(x_i, x_{i+1}), d(x_{i+2}, x_{i+3}) \geq \frac{k+2}{2} \quad and \quad d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2} \tag{23}$$

Let $x_i = v_a$, $x_{i+1} = v_b$, $x_{i+2} = v_c$, $x_{i+3} = v_d$. Since $diam(P_n^3) = k$, by equation (23) and our assumption that the sum of any pair of the distances, $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, $d(x_{i+2}, x_{i+3})$, is greater than $k+2$, it must be that $a < c < b < d$ (or $d < b < c < a$). Then

$$d(x_i, x_{i+3}) \geq d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) - 1.$$

So,

$$\begin{aligned} d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_{i+2}, x_{i+3}) &\leq d(x_i, x_{i+3}) + d(x_{i+1}, x_{i+2}) + 1 \\ &\leq d(x_i, x_{i+3}) + \frac{k+2}{2} + 1 \\ &= d(x_i, x_{i+3}) + \frac{k}{2} + 2 \end{aligned}$$

By equation **??**, we have

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k + 3 - d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) - d(x_{i+2}, x_{i+3}) \\ &\geq 3k + 3 - 2 - \frac{k}{2} - d(x_i, x_{i+3}) \\ &= k + 1 - d(x_i, x_{i+3}). \end{aligned}$$

Let $j \geq i+4$. Since $min\{d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2})\} \leq \frac{k+2}{2}$, and $f_i \geq k + 1 - d(x_i, x_{i+1})$ for any i, we have $max\{f_i, f_{i+1}\} \geq \frac{k}{2}$ for any $1 \leq i \leq n-2$. Hence,

$$\begin{aligned} f(x_j) - f(x_i) &\geq f_i + f_{i+1} + f_{i+2} + f_{i+3} \\ &\geq \left\{1 + \frac{k}{2}\right\} + \left\{1 + \frac{k}{2}\right\} \\ &> k + 1 > k + 1 - d(x_i, x_j) \end{aligned}$$

□

To show the existence of a radio-labeling achieving the desired bound, we consider cases separately. For each radio-labeling f given in the following, we shall first define a permutation (line-up) of the vertices $V(P_n^3) = \{x_1, x_2, \ldots, x_n\}$, then define f by $f(x_1) = 1$ and for $i = 1, 2, \ldots, n-1$:

$$f(x_{i+1}) = f(x_i) + diam(P_n^3) + 1 - d_{P_n^3}(x_i, x_{i+1}). \tag{24}$$

For the case $n \equiv 0 \pmod 6$

Let $n = 6p$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p \Rightarrow \frac{k}{2} = p$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_2, x_3 = v_{3p+3}, \ldots, x_n = v_{3p}$ as shown in the Table 1.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \le i \le n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 1 - d_{P_n^3}(x_i, x_{i+1}) \tag{25}$$

The function f defined in equation (25) chooses the vertices one from the right of the central vertex and other from the left for the consecutive labeling. The difference between two adjacent vertices in P_n is shown above the arrow. Since the minimum of any two consecutive distances is lesser than $3p+1$ and equal to $3p+1$ only if their sum is divisible by 3, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p-1]} v_2 \xrightarrow{[3p+1} v_{3p+3} \xrightarrow{3p-2} v_5 \xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p-2} v_8$$

$$\xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{6p-3} \xrightarrow{3p-2]} v_{3p-1} \xrightarrow{[3p+1]} v_{6p} \xrightarrow{[6p-1]} v_1$$

$$\xrightarrow{[3p+1} v_{3p+2} \xrightarrow{3p-2} v_4 \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p-2} v_7 \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1}$$

$$v_{6p-4} \xrightarrow{3p-2]} v_{3p-2} \xrightarrow{[3p+1]} v_{6p-1} \xrightarrow{[6p-4]} v_3 \xrightarrow{[3p+1} v_{3p+4} \xrightarrow{3p-2}$$

$$v_6 \xrightarrow{3p+1} v_{3p+7} \xrightarrow{3p-2} v_9 \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p-2]} v_{3p} = x_n$$

Table 1: A radio-labeling procedure for the graph P_n^3 when $n \equiv 0 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left\lceil \frac{3p-1}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil + \\
&\quad \left\lceil \frac{6p-1}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil + \\
&\quad \left\lceil \frac{6p-4}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) \\
&= p + (2p+1)(p-1) + (p+1) + 2p + (2p+1) + (p-1) + \\
&\quad (p+1) + (2p-1) + (2p+1)(p-1) \\
&= 6p^2 + 4p - 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\
&= (6p-1)(2p+1) - (6p^2 + 4p - 2) + 1 \\
&= 6p^2 + 2 = \frac{n^2}{2} + 2.
\end{aligned}$$

Hence,

$$rn(P_n^3) \leq \frac{n^2+12}{6}, \text{ if } n \equiv 0 \pmod 6$$

An example for this case is shown in Figure 9.

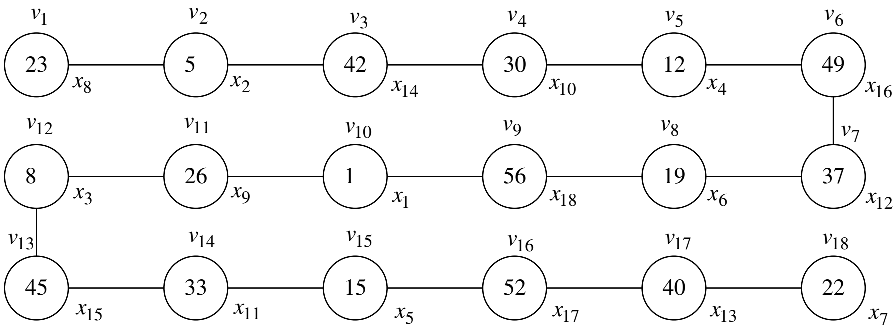


Figure 9: A minimal radio labeling of the graph P_{18}^3.

For the case $n \equiv 1 \pmod 6$

Let $n = 6p+1$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p \Rightarrow \frac{k}{2} = p$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_3, x_3 = v_{3p+4}, \ldots, x_n = v_{6p-1}$ as shown in the Table 2.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 1 - d_{P_n^3}(x_i, x_{i+1}). \tag{26}$$

For the function f defined in equation (26), The minimum difference between any two adjacent vertices in P_n is shown in Table 2 is less than $3p+1$ and equal to $3p+1$ only if their sum is divisible by 3, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p-2} v_3 \xrightarrow{3p+1} v_{3p+4} \xrightarrow{3p-2} v_6 \xrightarrow{3p+1} v_{3p+7} \xrightarrow{3p-2} v_9$$

$$\xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p-2} v_{3p} \xrightarrow{3p+1]} v_{6p+1} \xrightarrow{6p-1} v_2 \xrightarrow{[3p+1} v_{3p+3} \xrightarrow{3p-2}$$

$$v_5 \xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p-2} \bullet\bullet\bullet \xrightarrow{3p-2} v_{3p-1} \xrightarrow{3p+1]} v_{6p} \xrightarrow{[6p-1]} v_1 \xrightarrow{[3p+1}$$

$$v_{3p+2} \xrightarrow{3p-2} v_4 \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p-2} \bullet\bullet\bullet \xrightarrow{3p-2]} v_{3p-2} \xrightarrow{3p+1} v_{6p-1} = x_n$$

Table 2: A radio-labeling procedure for the graph P_n^3 when $n \equiv 1 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left\lceil \frac{3p-3}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil + \left(\left\lceil \frac{3p-2}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil \right)(p-1) + \left\lceil \frac{6p-4}{3} \right\rceil + \\ &\quad \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{6p-1}{3} \right\rceil + \\ &\quad \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{6p-1}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil \\ &= 2p + (2p+1)(p-2) + 2p - 1 + (2p+1)(p-1) + 3p + 1 + (2p+1)(p-1) + 4p + 2 \\ &= 6p^2 + 6p - 2.\end{aligned}$$

Therefore,

$$\begin{aligned} f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\ &= (6p)(2p+1) - (6p^2 + 6p - 2) + 1 \\ &= 6p^2 + 3 = \frac{n^2 - 2n + 19}{6}. \end{aligned}$$

Hence,

$$rn(P_n^3) \le \tfrac{n^2-2n+19}{6}, \text{ if } n \equiv 1 \pmod 6 \text{ and } n \ge 13$$

An example for this case is shown in Figure 10.

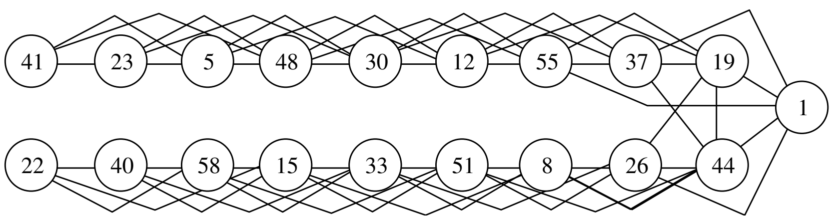


Figure 10: A minimal radio labeling of the graph P_{19}^3.

For the case $n \equiv 2 \pmod 6$

Let $n = 6p + 2$. Then $k = \left\lceil \frac{n-1}{3} \right\rceil = 2p + 1 \Rightarrow 3p + 2 < 3\frac{k}{2} + 1$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_{6p+2}, x_3 = v_{3p-2}, \ldots, x_n = v_{3p+3}$ as shown in the Table 3.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \le i \le n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 2 - d_{P_n^3}(x_i, x_{i+1}) \tag{27}$$

For the function f defined in equation (27), the minimum difference between any two adjacent vertices in P_n is shown in Table 3 is not greater than $3p+1$ and k is odd, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p+1} v_{6p+2} \xrightarrow{3p+4} v_{3p-2} \xrightarrow{3p+1} v_{6p-1} \xrightarrow{3p+4} v_{3p-5} \xrightarrow{3p+1}$$

$$v_{6p-4} \xrightarrow{3p+4} \bullet\bullet\bullet \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p+4]} v_1 \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[2]} v_{3p}$$

$$\xrightarrow{[3p+1} v_{6p+1} \xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6} \xrightarrow{3p+1} v_{6p-5}$$

$$\xrightarrow{3p+4} v_{3p-9} \bullet\bullet\bullet \xrightarrow{3p+4]} v_3 \xrightarrow{[3p+1]} v_{3p+4} \xrightarrow{[5]} v_{3p-1} \xrightarrow{[3p+1} v_{6p}$$

$$\xrightarrow{3p+4} v_{3p-4} \xrightarrow{3p+1} v_{6p-3} \xrightarrow{3p+4} v_{3p-7} \xrightarrow{3p+1} v_{6p-6} \bullet\bullet\bullet \xrightarrow{3p+4} v_5$$

$$\xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p+4]} v_2 \xrightarrow{3p+1} v_{3p+3} = x_n$$

Table 3: A radio-labeling procedure for the graph P_n^3 when $n \equiv 2 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p+4}{3}\right\rceil\right)(p) + \left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{2}{3}\right\rceil + \\
&\quad \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p+4}{3}\right\rceil\right)(p-1) + \left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{5}{3}\right\rceil + \\
&\quad \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p+4}{3}\right\rceil\right)(p-1) + \left\lceil\frac{3p+1}{3}\right\rceil \\
&= (2p+3)(p) + (p+1) + 1 + (2p+3)(p-1) + (p+1) + \\
&\quad 2 + (2p+3)(p-1) + (p+1) \\
&= 6p^2 + 8p.
\end{aligned}$$

Therefore,

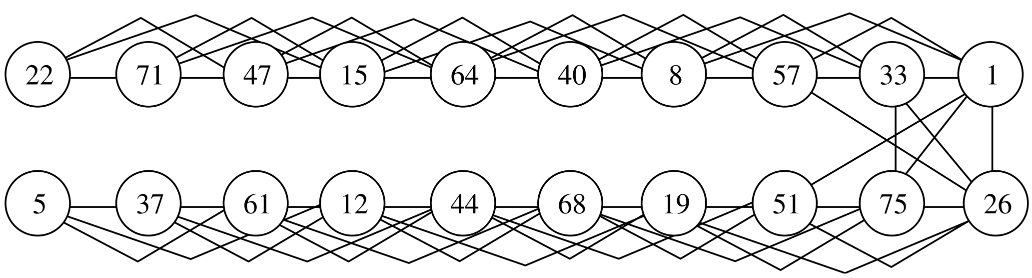


Figure 11: A minimal radio labeling of the graph P_{20}^3.

$$\begin{aligned} f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\ &= (6p+1)(2p+2) - (6p^2 + 8p) + 1 \\ &= 6p^2 + p + 3 = \frac{n^2 + 2n + 10}{6}. \end{aligned}$$

Hence,

$$rn(P_n^3) \leq \tfrac{n^2+2n+10}{6}., \text{ if } n \equiv 2 \pmod 6$$

An example for this case is shown in Figure 11.

For the case $n \equiv 3$ (mod 6)

Let $n = 6p+3$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p+1 \Rightarrow 3p+2 < 3\frac{k}{2} + 1$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_{6p+2}, x_3 = v_1, \ldots, x_n = v_{3p+4}$ as shown in the Table 4.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 2 - d_{P_n^3}(x_i, x_{i+1}) \tag{28}$$

For the function f defined in equation (28), the minimum difference between any two adjacent vertices in P_n is shown in Table 4 is not greater than $3p+1$ and k is odd, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p+1]} v_{6p+2} \xrightarrow{[6p+1]} v_1 \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[3p+1]} v_{6p+3} \xrightarrow{[6p+1]}$$

$$v_2 \xrightarrow{[3p+1} v_{3p+3} \xrightarrow{3p-2} v_5 \xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p-2} v_8 \xrightarrow{3p+1} v_{3p+9} \bullet\bullet\bullet$$

$$v_{6p-3} \xrightarrow{3p-2]} v_{3p-1} \xrightarrow{[3p+1]} v_{6p} \xrightarrow{[6p-4]} v_4 \xrightarrow{[3p+1} v_{3p+5} \xrightarrow{3p-2} v_7$$

$$\xrightarrow{3p+1} v_{3p+8} \xrightarrow{3p-2} \bullet\bullet\bullet\ v_{6p-4} \xrightarrow{3p-2]} v_{3p-2} \xrightarrow{[3p+1]} v_{6p-1} \xrightarrow{[3p-1]}$$

$$v_{3p} \xrightarrow{[3p+1} v_{6p+1} \xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6} \xrightarrow{3p+1}$$

$$v_{6p-5} \bullet\bullet\bullet\ v_{3p+7} \xrightarrow{3p+4]} v_3 \xrightarrow{[3p+1]} v_{3p+4} = x_n$$

Table 4: A radio-labeling procedure for the graph P_n^3 when $n \equiv 3$ (mod 6).

For the labeling f defined above we get

$$\begin{aligned}\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{6p+1}{3}\right\rceil + \left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{6p+1}{3}\right\rceil + \\ &\quad \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p-2}{3}\right\rceil\right)(p-1) + \left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{6p-4}{3}\right\rceil + \\ &\quad \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p-2}{3}\right\rceil\right)(p-2) + \left\lceil\frac{3p+1}{3}\right\rceil + \\ &\quad \left\lceil\frac{3p-1}{3}\right\rceil + \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p-4}{3}\right\rceil\right)(p-1) + \left\lceil\frac{3p+1}{3}\right\rceil \\ &= (p+1) + (2p+1) + (p+1) + (p+1)(2p+1) + \\ &\quad (2p+1)(p-1) + (p+1) + (2p-1) + \\ &\quad (2p+1)(p-2) + (p+1) + p + (2p+3)(p-1) + (p+1) \\ &= 6p^2 + 10p + 1.\end{aligned}$$

Therefore,

$$\begin{aligned}f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\ &= (6p+2)(2p+2) - (6p^2 + 10p + 1) + 1 \\ &= 6p^2 + 6p + 4 = \frac{n^2 + 15}{6}.\end{aligned}$$

Hence,

$$rn(P_n^3) \le \tfrac{n^2+15}{6}, \text{ if } n \equiv 3 \pmod 6$$

An example for this case is shown in Figure 12.

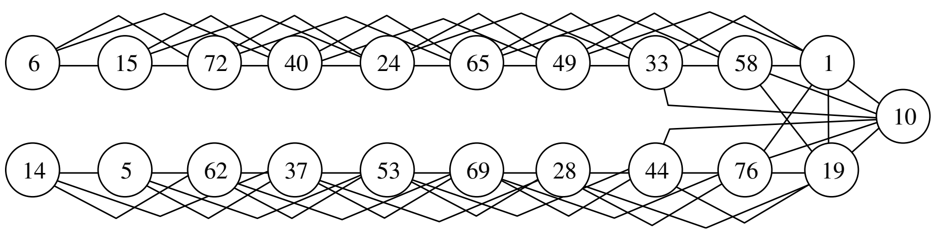


Figure 12: A minimal radio labeling of the graph P_{21}^3

For the case $n \equiv 4$ (mod 6)

Let $n = 6p + 4$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p + 1 \Rightarrow 3p + 2 \le 3\frac{k}{2} + 1$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_{6p+2}, x_3 = v_{3p-2}, \ldots, x_n = v_{6p+4}$ as shown in the Table 5.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \le i \le n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 2 - d_{P_n^3}(x_i, x_{i+1}). \tag{29}$$

For the function f defined in equation (29), the minimum difference between any two adjacent vertices in P_n is shown in Table 5 is not greater than $3p+4$ and k is odd, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p+1]} v_{6p+2} \xrightarrow{[3p+4]} v_{3p-2} \xrightarrow{[3p+1} v_{6p-1} \xrightarrow{3p+4} v_{3p-5}$$

$$\xrightarrow{3p+1} v_{6p-4} \xrightarrow{3p+4} v_{3p-8} \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p+4]} v_1$$

$$\xrightarrow{[3p+2]} v_{3p+3} \xrightarrow{[3p+1} v_2 \xrightarrow{3p+4} v_{3p+6} \xrightarrow{3p+1} v_5 \xrightarrow{3p+4} \bullet\bullet\bullet$$

$$\xrightarrow{3p+4} v_{6p} \xrightarrow{3p+1} v_{3p-1} \xrightarrow{3p+1]} v_{6p+3} \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[3p+2]} v_{6p+4}$$

$$\xrightarrow{[3p+4} v_{3p} \xrightarrow{3p+1} v_{6p+1} \xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6}$$

$$\xrightarrow{3p+1} \bullet\bullet\bullet\ v_{3p+7} \xrightarrow{3p+4} v_3 \xrightarrow{3p+1]} v_{3p+4} = x_n$$

Table 5: A radio-labeling procedure for the graph P_n^3 when $n \equiv 4 \pmod 6$.

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil \right)(p-1) + \\
&\quad \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil \right)(p) + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+2}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil \right)(p) \\
&= (p+1) + (p+2) + (2p+3)(p-1) + (p+1) + (2p+3)p + \\
&\quad (p+1) + (p+1) + (2p+3)p = 6p^2 + 12p + 3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\
&= (6p+3)(2p+2) - (6p^2 + 12p + 3) + 1 \\
&= 6p^2 + 6p + 4 = \frac{n^2 - 2n + 16}{6}.
\end{aligned}$$

Hence,

$$rn(P_n^3) \leq \tfrac{n^2-2n+16}{6}, \text{ if } n \equiv 4 \pmod 6$$

An example for this case is shown in Figure 13.

For the case $n \equiv 5 \pmod 6$

Let $n = 6p+5$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p+2 \Rightarrow 3p+4 \leq 3\frac{k}{2} + 1$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+3}, x_2 = v_2, x_3 = v_{3p+6}, \ldots, x_n = v_{3p+4}$ as shown in the Table 6.

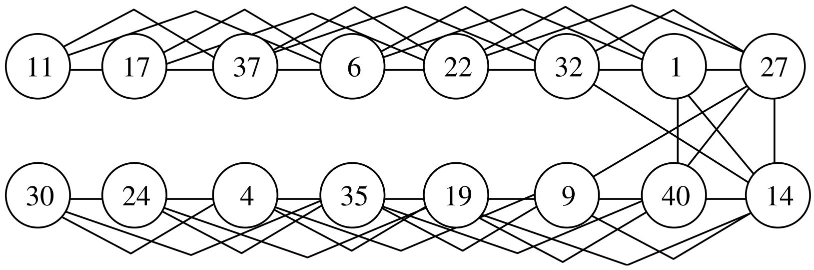


Figure 13: A minimal radio labeling of the graph P_{16}^3.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 3 - d_{P_n^3}(x_i, x_{i+1}) \tag{30}$$

For the function f defined in equation (30), the maximum difference between any two adjacent vertices in P_n is shown in Table 6 is less than or equal to $3p+4$ and the equality holds only if their sum is divisible by 3, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+3} \xrightarrow{[3p+1} v_2 \xrightarrow{3p+4} v_{3p+6} \xrightarrow{3p+1} v_5 \xrightarrow{3p+4} v_{3p+9} \xrightarrow{3p+1} \bullet\bullet\bullet$$

$$\xrightarrow{3p+1} v_{3p-1} \xrightarrow{3p+4]} v_{6p+3} \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[3p+3]} v_{6p+5} \xrightarrow{[3p+4} v_{3p+1}$$

$$\xrightarrow{3p+1} v_{6p+2} \xrightarrow{3p+4} v_{3p-2} \xrightarrow{3p+1} v_{6p-1} \xrightarrow{3p+4} \bullet\bullet\bullet \xrightarrow{3p+4} v_4$$

$$\xrightarrow{3p+1]} v_{3p+5} \xrightarrow{[3p+4]} v_1 \xrightarrow{[6p+3]} v_{6p+4} \xrightarrow{[3p+4} v_{3p} \xrightarrow{3p+1} v_{6p+1}$$

$$\xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6} \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{3p+7}$$

$$\xrightarrow{3p+4} v_3 \xrightarrow{3p+1]} v_{3p+4} = x_n$$

Table 6: A radio-labeling procedure for the graph P_n^3 when $n \equiv 5 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil\right)(p) + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+3}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil\right)(p) + \left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{6p+3}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil\right)(p) \\
&= (2p+3)p + (p+1) + (p+1) + (2p+1)p + (p+2) + \\
&\quad (2p+1) + (2p+3)(p) = 6p^2 + 14p + 5.
\end{aligned}$$

Therefore,

$$\begin{aligned} f(x_n) &= (n-1)(diamP_2^3+1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\ &= (6p+4)(2p+3) - (6p^2+14p+5) + 1 \\ &= 6p^2+12p+8 = \frac{n^2+2n+13}{6}. \end{aligned}$$

Hence,

$$rn(P_n^3) \leq \tfrac{n^2+2n+13}{6}, \text{ if } n \equiv 5 \pmod 6$$

An example for this case is shown in Figure 14.

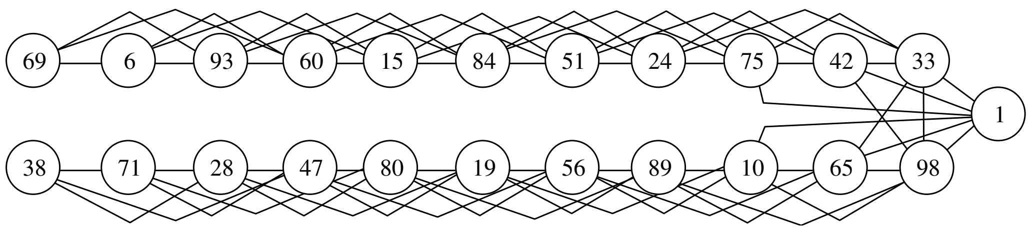


Figure 14: A minimal radio labeling of the graph P_{23}^3.

§4. Radio labeling of P_n^3 for $n \leq 5$ or $n = 7$

In this section we determine radio numbers of cube path of small order as a special case.

Theorem 4.1 *For any integer n, $1 \leq n \leq 5$, the radio number of the graph P_n^3 is given by*

$$rn(P_n^2) = \begin{cases} n, & if\ n = 1,2,3,4 \\ 8, & if\ n = 5,7 \end{cases}$$

Proof If $n \leq 4$, the graph is isomorphic to K_n and hence the result follows immediately. Now consider the case $n = 5$, we see that there is exactly one pair of vertices at a distance 2 and all other pairs are adjacent, so maximum value of $\sum_1^4 d(x_i, x_{i+1}) = 2+1+1+1 = 5$.

Now, consider a radio labeling f of P_5^3 and label the vertices as x_1, x_2, x_3, x_4, x_5 such that $f(x_i) < f(x_{i+1})$, then

$$\begin{aligned} f(x_n) - f(x_1) &\geq (n-1)(diamP_5^3+1) - \sum_{1}^{4} d(x_i, x_{i+1}) \\ &\geq 4(3) - 5 = 7 \\ \Rightarrow f(x_n) &\geq 7 + f(x_1) = 8 \end{aligned}$$

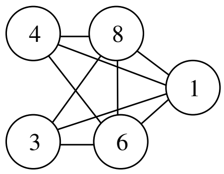


Figure 15: A minimal radio labeling of P_5^3.

includegraphics[width=8cm]figlast2.eps

Figure 16: A minimal radio labeling of P_7^3

On the other hand, In the Figure 15, we verify that the labels assigned for the vertices serve as a radio labeling with span 8, so $rn(P_n^3) = 8$.

, similarly if $n = 7$, then, as the central vertex of P_n^3 is adjacent to every other vertex, maximum value of $\sum^6 i = 1d(x_i, x_{i+1}) = 2 \times 5 + 1 = 11$. So, as above, $f(x_n) \geq (6)(2 + 1) - 11 + 1 = 8$. The reverse inequality follows by the Figure 16. Hence the theorem. □

Acknowledgment

Authors are very much thankful to the referees for their valuable suggestions for the improvement of the paper and the Principals Prof. Martin Jebaraj of Dr. Ambedkar Institute of Technology Prof. Rana Prathap Reddy of Reva Institute of Technology and Prof. K. R. Suresh of Bangalore Institute of Technology, for their constant support and encouragement during the preparation of this paper.

References

[1] Buckley F and Harary F, *Distance in Graphs*, Addison-Wesley,(1990).

[2] G. Chartrand, D. Erwin, F. Harary and P. Zhang, *Radio labelings of graphs*, Bull. Inst. Combin. Appl., 33 (2001), 77-85.

[3] G. Chartrand, D. Erwin, F. Harary and P. Zhang, *Radio Antipodal Colorings of Cycles*, Congr. Numer, 144 (2000),129-141.

[4] Gary Chartrand, David Erwin, Ping Zhang, *A graph labeling problem Suggested by FM Channel Restrictions*, Bulletin of the Inst. Combin. Appl. 43 (2005), 43-57.

[5] D. Liu and M. Xie, *Radio number for square of cycles*, congr, Numer., 169 (2004) 105-125.

[6] D.D.F. Liu and M. Xie, *Radio number of square paths*, Ars Comb., 90(2009), 307-319.

[7] Daphne Der-Fen Liu and Xuding Zhu, *Multilevel Distance Labelings for paths and cycles*, SIAM J. Discrete Math., 19 (2005) 610-621.

[8] Daphne Der-Fen Liu, *Radio Number for Trees*, preprint 2006 (to appear)

[9] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, # DS6,(2009),1-219.

[10] W.K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE 68 (1980), 1497-1514.

[11] Hartsfield Gerhard and Ringel, *Pearls in Graph Theory*, Academic Press, USA, 1994.

[12] B. Sooryanarayana and Raghunath.P, *Radio labeling of cube of a cycle*, Far East J. Appl. Math 29(1)(2007), 113-147.

[13] B. Sooryanarayana and Raghunath.P, *Radio labeling of* C_n^4, Journal of Applied Mathematical Analysis and Applications 3(2007) no. 2, 195-228.

[14] J.Van den Heuvel, R.Leese and M.Shepherd, *Graph labeling and radio Channel Assignment*, J.Graph Theory, 29(1998), 263-283.

[15] P.Zhang, *Radio labellings of Cycles*, Ars Combin., 65 (2002), 21-32.