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Further Results on Product Cordial Labeling

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Abstract: We prove that closed helm CH_n, web graph Wb_n, flower graph Fl_n, double triangular snake DT_n and gear graph G_n admit product cordial labeling.

Key Words: Graph labeling, cordial labeling, Smarandachely p-product cordial labeling, product cordial labeling.

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§1. Introduction

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For any undefined notations and terminology we rely upon Clark and Holton [3]. In order to maintain compactness we provide a brief summery of definitions and existing results.

Definition 1.1 *A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).*

According to Beineke and Hegde [1] labeling of discrete structure serves as a frontier between graph theory and theory of numbers. A dynamic survey of graph labeling is carried out and frequently updated by Gallian [4].

Definition 1.2 *A mapping $f : V(G) \to \{0,1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

The induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let us denote $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

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Definition 1.3 *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called cordial if it admits cordial labeling.*

The concept of cordial labeling was introduced by Cahit [2] in which he investigated several results on this newly defined concept. After this some labelings like prime cordial labeling, A - cordial labeling, H-cordial labeling and product cordial labeling are also introduced as variants of cordial labeling.

This paper is aimed to report some new families of product cordial graphs.

Definition 1.4 *For an integer $p > 1$. A mapping $f : V(G) \to \{0, 1, 2, \cdots, p\}$ is called a Smarandachely p-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2, \cdots, p-1\}$, where $v_f(i)$ denotes the number of vertices labeled with i, $e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i(mod\ p)$. Particularly, if $p = 2$, i.e., a binary vertex labeling of graph G with an induced edge labeling $f^* : E(G) \to \{0, 1\}$ defined by $f^*(e = uv) = f(u)f(v)$, such a Smarandachely 2-product cordial labeling is called product cordial labeling. A graph with product cordial labeling is called a product cordial graph.*

The product cordial labeling was introduced by Sundaram et al. [5] and they investigated several results on this newly defined concept. They have established a necessary condition showing that a graph with p vertices and q edges with $p \geq 4$ is product cordial then $q < (p^2 - 1)/4 + 1$.

The graphs obtained by joining apex vertices of k copies of stars, shells and wheels to a new vertex are proved to be product cordial by Vaidya and Dani [6] while some results on product cordial labeling for cycle related graphs are reported in Vaidya and Kanani [7].

Vaidya and Barasara [8] have proved that the cycle with one chord, the cycle with twin chords, the friendship graph and the middle graph of path admit product cordial labeling. The same authors in [9] have proved that the graphs obtained by duplication of one edge, mutual vertex duplication and mutual edge duplication in cycle are product cordial graphs. Vaidya and Vyas [10] have discussed product cordial labeling in the context of tensor product of some graphs while Vaidya and Barasara [11] have investigated some results on product cordial labeling in the context of some graph operations.

Definition 1.5 *The wheel graph W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as apex vertex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges. We continue to recognize apex of respective graphs obtained from wheel in Definitions 1.6 to 1.9.*

Definition 1.6 *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 1.7 *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.*

Definition 1.8 *The web graph Wb_n is the graph obtained by joining the pendant vertices of a helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle.*

Definition 1.9 *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.*

Definition 1.10 *The double triangular snake DT_n is obtained from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \cdots, n-1$ and to a new vertex u_i for $i = 1, 2, \cdots, n-1$.*

Definition 1.11 *Let $e = uv$ be an edge of graph G and w is not a vertex of G. The edge e is subdivided when it is replaced by edges $e' = uw$ and $e'' = wv$.*

Definition 1.12 *The gear graph G_n is obtained from the wheel by subdividing each of its rim edge.*

§2. Main Results

Theorem 2.1 *Closed helm CH_n is a product cordial graph.*

Proof Let v be the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of inner cycle and $u_1, u_2, \ldots, u_n$ be the vertices of outer cycle of CH_n. Then $|V(CH_n)| = 2n+1$ and $|E(CH_n)| = 4n$.

We define $f : V(CH_n) \to \{0,1\}$ to be $f(v) = 1,\ f(v_i) = 1$ and $f(u_i) = 0$ for all i. In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = n$, $e_f(0) = e_f(1) = 2n$. Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence CH_n is a product cordial graph. □

Illustration 2.2 The Fig.1 shows the closed helm CH_5 and its product cordial labeling.

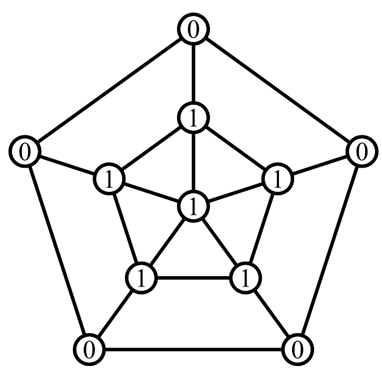

Fig.1

Theorem 2.3 *Web graph Wb_n admits product cordial labeling.*

Proof Let v be the apex vertex, $v_1, v_2, \cdots, v_n$ be the vertices of inner cycle, $v_{n+1}, v_{n+2}, \cdots, v_{2n}$ be the vertices of outer cycle and $v_{2n+1}, v_{2n+2}, \cdots, v_{3n}$ be the pendant vertices in Wb_n. Then $|V(Wb_n)| = 3n+1$ and $|E(Wb_n)| = 5n$.

To define $f : V(Wb_n) \to \{0,1\}$ we consider following two cases.

Case 1. n is odd

Define $f(v) = 1$, $f(v_i) = 1$ for $1 \leq i \leq n$, $f(v_{2i}) = 1$ for $\left\lceil \frac{n}{2} \right\rceil \leq i \leq n-1$ and $f(v_i) = 0$ otherwise. In view of the above labeling patten we have $v_f(0) = v_f(1) = \frac{3n+1}{2}$, $e_f(0) - 1 = e_f(1) = \frac{5n-1}{2}$.

Case 2. n is even

Define $f(v) = 1$, $f(v_i) = 1$ for $1 \leq i \leq n$, $f(v_{2i+1}) = 1$ for $\frac{n}{2} \leq i \leq n-1$ and $f(v_i) = 0$ otherwise. In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = \frac{3n}{2}$, $e_f(0) = e_f(1) = \frac{5n}{2}$. Thus in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence Wb_n admits product cordial labeling. □

Illustration 2.4 The Fig.2 shows the web graph Wb_5 and its product cordial labeling.

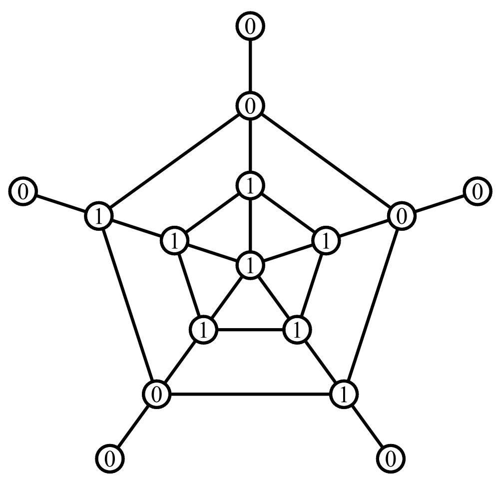

Fig.2

Theorem 2.5 *Flower graph Fl_n admits product cordial labeling.*

Proof Let H_n be a helm with v as the apex vertex, $v_1, v_2, \cdots, v_n$ be the vertices of cycle and $v_{n+1}, v_{n+2}, \cdots, v_{2n}$ be the pendant vertices. Let Fl_n be the flower graph obtained from helm H_n. Then $|V(Fl_n)| = 2n+1$ and $|E(Fl_n)| = 4n$.

We define $f : V(Fl_n) \to \{0, 1\}$ to be $f(v) = 1$, $f(v_i) = 1$ for $1 \leq i \leq n$ and $f(v_i) = 0$ for $n+1 \leq i \leq 2n$. In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = n$, $e_f(0) = e_f(1) = 2n$. Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence Fl_n admits product cordial labeling. □

Illustration 2.6 The Fig.3 shows flower graph Fl_5 and its product cordial labeling.

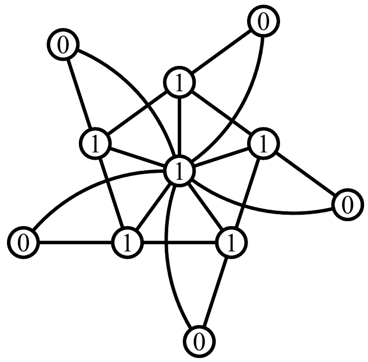

Fig.3

Theorem 2.7 *Double triangular snake DT_n is a product cordial graph for odd n and not a product cordial graph for even n.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of path P_n and $v_{n+1}, v_{n+2}, \cdots, v_{3n-2}$ be the newly added vertices in order to obtain DT_n. Then $|V(DT_n)| = 3n-2$ and $|E(DT_n)| = 5n-5$.

To define $f : V(DT_n) \to \{0,1\}$ we consider following two cases.

Case 1. n is odd

$f(v_i) = 0$ for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$, $f(v_i) = 0$ for $n+1 \le i \le n + \left\lfloor \frac{n}{2} \right\rfloor$ and $f(v_i) = 1$ otherwise. In view of the above labeling patten we have $v_f(0) + 1 = v_f(1) = \left\lceil \frac{3n-2}{2} \right\rceil$, $e_f(0) - 1 = e_f(1) = \frac{5n-5}{2}$. Thus we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

Case 2. n is even

Subcase 1. $n = 2$.

The graph DT_2 has $p = 4$ vertices and $q = 5$ edges since

$$\frac{p^2-1}{4} + 1 = \frac{19}{4} < q.$$

Thus the necessary condition for product cordial graph is violated. Hence DT_2 is not a product cordial graph.

Subcase 2. $n \ne 2$

In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\frac{3n-2}{2}$ vertices out of $3n-2$ vertices. The vertices with label 0 will give rise at least $\frac{5n}{2} - 1$ edges with label 0 and at most $\frac{5n}{2} - 4$ edge with label 1 out of total $5n-5$ edges. Therefore $|e_f(0) - e_f(1)| = 3$. Thus the edge condition for product cordial graph is violated. Therefore DT_n is not a product cordial graph for even n.

Hence Double triangular snake DT_n is a product cordial graph for odd n and not a product cordial graph for even n. □

Illustration 2.8 The Fig.4 shows the double triangular snake DT_7 and its product cordial labeling.

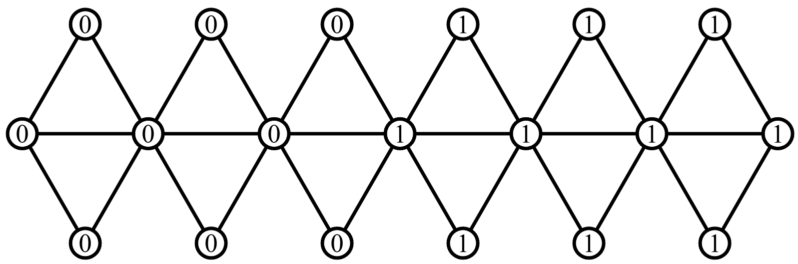

Fig.4

Theorem 2.9 *Gear graph G_n is a product cordial graph for odd n and not product cordial graph for even n.*

Proof Let W_n be the wheel with apex vertex v and rim vertices $v_1, v_2, \cdots, v_n$. To obtain the gear graph G_n subdivide each rim edge of wheel by the vertices $u_1, u_2, \cdots, u_n$. Where each u_i subdivides the edge v_iv_{i+1} for $i = 1, 2, \cdots, n-1$ and u_n subdivides the edge v_1v_n. Then $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$.

To define $f : V(G_n) \to \{0, 1\}$ we consider following two cases.

Case 1. n is odd

$f(v) = 1$; $f(v_i) = 1$ for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$; $f(v_i) = 0$, otherwise;

$f(u_i) = 1$ for $1 \leq i \leq n + \left\lfloor \frac{n}{2} \right\rfloor$; $f(u_i) = 0$, otherwise.

In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = n$, $e_f(0) = e_f(1) + 1 = \frac{3n+1}{2}$. Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2. n is even

In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to n vertices out of $2n + 1$ vertices. The vertices with label 0 will give rise at least $\frac{3n}{2} + 1$ edges with label 0 and at most $\frac{3n}{2} - 1$ edge with label 1 out of total $3n$ edges. Therefore $|e_f(0) - e_f(1)| = 2$. Thus the edge condition for product cordial graph is violated. So G_n is not a product cordial graph for even n.

Hence gear graph is a product cordial graph for odd n and not product cordial graph for even n. □

Illustration 2.10 The Fig.5 shows the gear graph G_7 and its product cordial labeling.

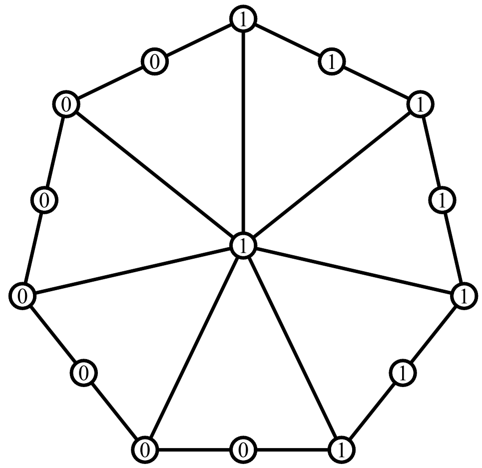

Fig.5

§3. Concluding Remarks

Some new families of product cordial graphs are investigated. To investigate some characterization(s) or sufficient condition(s) for the graph to be product cordial is an open area of research.

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