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Operator Exponentials for the Clifford Fourier Transform on Multivector Fields

by

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Abstract

This paper briefly reviews the notion of Clifford's geometric algebras and vector to multivector functions; as well as the field of Clifford analysis (function theory of the Dirac operator). In Clifford Fourier transformations (CFT) on multivector signals the complex unit $i \in \mathbb{C}$ is replaced by a multivector square root of -1, which may be a pseudoscalar in the simplest case. For these transforms we derive, via a multivector function representation in terms of monogenic polynomials, the operator representation of the CFTs by exponentiating the Hamilton operator of a harmonic oscillator.

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§1. Introduction

The Clifford Fourier transform we refer to was originally introduced by B. Jancewicz [20] for electro-magnetic field computations in Clifford's geometric algebra $\mathcal{G}_3 = Cl(3,0)$ of $\mathbb{R}^3$, replacing the imaginary complex unit $i \in \mathbb{C}$ by the central unit pseudoscalar $i_3 \in \mathcal{G}_3$, which squares to minus one. This type of CFT was subsequently expanded to $\mathcal{G}_2$, instead using $i_2 \in \mathcal{G}_2$, and applied to image structure computations by M. Felsberg [13]. J. Ebling and G. Scheuermann [12] applied both these CFTs to the study of vector fields, as they occur in two and three dimensional physical flows. E. Hitzer and B. Mawardi [19] extended these CFTs to higher dimensions $n = 2, 3 (\text{mod } 4)$ and studied their properties, including the physical uncertainty principle for multivector fields.

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Independently De Bie et al [9] showed how Fourier transforms can be generalized to Clifford algebras, by introducing an operator form for the complex Fourier transform and generalizing it in the framework of Clifford analysis. This approach relies on the existence of a particular realization for the Lie algebra $\mathfrak{sl}(2)$, closely connected to the Hamilton operator of a harmonic oscillator, which makes it possible to introduce a fractional Clifford Fourier transform (see [7]) and to study radial deformations (see [8, 6])

In the present paper we show how the Clifford Fourier transforms (CFT) of [19] on multivector fields in $\mathcal{S}(\mathbb{R}^n, \mathcal{G}_n)$ can be written in the form of an *exponential of the Hamilton operator of the harmonic oscillator* in n dimensions and a constant phase factor depending on the dimension. For the proof of this fundamental result the properties of monogenic functions of degree k and of Clifford-Hermite functions as an intermediate function representation, play a crucial role. The computation of an integral transformation is thus replaced by the application of a differential operator, which has profound physical meaning. We therefore expect our result to be valuable not only as another way to represent and compute the CFT, but beyond its mathematical aspect to shed new light on the closely related roles of the harmonic oscillator and the CFT in nature, in particular in physics, and in its wide field of technical applications.

We note that in quantum physics the Fourier transform of a wave function is called the momentum representation, and that the multiplication of the wave function with the exponential of the Hamilton operator times time represents the transition between the Schrödinger- and the Heisenberg representations [22]. The fact that the exponential of the Hamilton operator can also produce the change from position to momentum representation in quantum mechanics adds a very interesting facet to the picture of quantum mechanics.

The paper is structured as follows. Section 2 introduces the notion of Clifford's geometric algebras and basic methods of computation with vectors, multivectors (general elements of a geometric algebra) and functions mapping vectors to multivectors. Section 3 introduces the field of Clifford analysis (a function theory for the Dirac operator). Section 4 reviews the notion of Clifford Fourier transformations (CFT) and derives the operator representation of the CFTs by exponentiating the Hamilton operator of a harmonic oscillator, hereby using a multivector function representation in terms of monogenic polynomials.

§2. Clifford's Geometric Algebra $\mathcal{G}_n$

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ denote an orthonormal basis for the real n-dimensional Euclidean vector space $\mathbb{R}^n$ with $n = 2, 3 \pmod 4$. The geometric algebra over $\mathbb{R}^n$, denoted by means of $\mathcal{G}_n = Cl(n,0)$, then has the graded 2^n-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \dots, i_n = \boldsymbol{e}_1\boldsymbol{e}_2 \dots \boldsymbol{e}_n\}. \tag{2.1}$$

The element i_n has a special meaning, as it is the so-called pseudo-scalar. This element of the algebra $\mathcal{G}_n$ will play a crucial role in this paper (see below).

Remark 2.1. The fact that we begin by introducing orthonormal bases for both the vector space $\mathbb{R}^n$ and for its associated geometric algebra $\mathcal{G}_n$ is only because we assume readers to be familiar with these concepts. As is well-known, the definitions of vector spaces and geometric algebras are generically basis independent [16]. The definition of the vector derivative of section 2.1 is basis independent, too. Only when we introduce the infinitesimal scalar volume element for integration over $\mathbb{R}^n$ in section 4 we do express it with the help of a basis for the sake of computation. All results derived in this paper are therefore *manifestly invariant* (independent of the choice of coordinate systems).

The squares of vectors are positive definite scalars (Euclidean metric) and so are all even powers of vectors:

$$\boldsymbol{x}^2 \geq 0 \;\Rightarrow\; \boldsymbol{x}^m \geq 0 \quad \text{for} \quad m = 2m' \; (m' \in \mathbb{N}) \; . \tag{2.2}$$

Therefore, given a multivector $M \in \mathcal{G}_n$ one has:

$$\boldsymbol{x}^m M = M\,\boldsymbol{x}^m \quad \text{for} \quad m = 2m' \; (m' \in \mathbb{N}) \; . \tag{2.3}$$

Note that for $n = 2, 3 \,(\mathrm{mod}\, 4)$ one has that

$$i_n^2 = -1, \quad i_n^{-1} = -i_n, \quad i_n^m = (-1)^{\frac{m}{2}} \quad \text{for} \quad m = 2m' \; (m' \in \mathbb{Z}) \; . \tag{2.4}$$

similar to the complex imaginary unit. The *grade selector* is defined as $\langle M \rangle_k$ for the k-vector part of $M \in \mathcal{G}_n$, especially $\langle M \rangle = \langle M \rangle_0$. This means that each M can be expressed as the sum of all its grade parts:

$$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n. \tag{2.5}$$

The *reverse* of $M \in \mathcal{G}_n$ is defined by the anti-automorphism

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k. \tag{2.6}$$

The *square norm* of M is defined by

$$\|M\|^2 = \langle M\widetilde{M} \rangle \; , \tag{2.7}$$

which can also be expressed as $M * \widetilde{M}$, in terms of the real-valued (scalar) inner product

$$M * \widetilde{N} := \langle M\widetilde{N}\rangle \ . \tag{2.8}$$

Remark 2.2. For vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n \subset \mathcal{G}_n$ the inner product is identical with the scalar product (2.8),

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} * \widetilde{\boldsymbol{b}} = \boldsymbol{a} * \boldsymbol{b} \, . \tag{2.9}$$

As a consequence we obtain the *multivector Cauchy-Schwarz inequality*

$$|\langle M\widetilde{N}\rangle|^2 \ \leq \ \|M\|^2 \, \|N\|^2 \quad \forall \, M, N \in \mathcal{G}_n. \tag{2.10}$$

§2.1. Multivector Functions, Vector Differential and Vector Derivative

Let $f = f(\boldsymbol{x})$ be a multivector-valued function of a vector variable $\boldsymbol{x}$ in $\mathbb{R}^n$. For an arbitrary vector $\boldsymbol{a} \in \mathbb{R}^n$ we then define[1] the *vector differential* in the $\boldsymbol{a}$ direction as

$$\boldsymbol{a} \cdot \nabla f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{f(\boldsymbol{x} + \epsilon \boldsymbol{a}) - f(\boldsymbol{x})}{\epsilon} \tag{2.11}$$

provided this limit exists and is well defined.

Remark 2.3. For all $\boldsymbol{a} \in \mathbb{R}^n$, the operator $\boldsymbol{a} \cdot \nabla$ is a scalar operator, which means that the left and right vector differentials[2] coincide, i.e.

$$\boldsymbol{a} \cdot \dot{\nabla} \dot{f}(\boldsymbol{x}) = \dot{f}(\boldsymbol{x}) \, \boldsymbol{a} \cdot \dot{\nabla}. \tag{2.12}$$

Whenever there is not danger of ambiguity, the overdots in (2.12), (2.16), etc., may also be omitted.

The basis independent *vector derivative* ∇ is defined in [17, 18, 16] to have the algebraic properties[3] of a grade one vector in $\mathbb{R}^n$ and to obey equation (2.11) for all vectors $\boldsymbol{a} \in \mathbb{R}^n$. This allows the following explicit representation.

[1] Bracket convention: $A \cdot BC = (A \cdot B)C \neq A \cdot (BC)$ and $A \wedge BC = (A \wedge B)C \neq A \wedge (BC)$ for multivectors $A, B, C \in \mathcal{G}_n$. The vector variable index $\boldsymbol{x}$ of the vector derivative is dropped: $\boldsymbol{\nabla_x} = \boldsymbol{\nabla}$ and $\boldsymbol{a} \cdot \boldsymbol{\nabla_x} = \boldsymbol{a} \cdot \boldsymbol{\nabla}$, but not when differentiating with respect to a different vector variable (compare e.g. proposition 2.7).

[2] The Hestenes' overdot symbols specify on which function the vector derivative is supposed to act. Conventionally an operator applies to everything on its right, but the overdot notation can also show application to functions on the left side of an operator. Since algebraically scalars commute with all multivectors, the scalar character of the operator $\boldsymbol{a} \cdot \nabla$ then ensures that the right side of (2.12) is identical to the left side.

[3] In particular see e.g. proposition 18 of [17]. Based on these properties the full meaning of propositions 2.7 and 2.8 in the current paper can be understood.

Remark 2.4. The vector derivative ∇ can be expanded in a basis of $\mathbb{R}^n$ as

$$\nabla = \sum_{k=1}^{n} \boldsymbol{e}_k \partial_k \quad \text{with} \quad \partial_k = \frac{\partial}{\partial x_k}\,, \;\; 1 \leq k \leq n. \tag{2.13}$$

Proposition 2.5 (Left and right linearity).

$$\nabla(f+g) = \nabla f + \nabla g\,, \quad (f+g)\nabla = f\nabla + g\nabla\,. \tag{2.14}$$

Proposition 2.6. For $f(\boldsymbol{x}) = g(\lambda(\boldsymbol{x}))$, $\lambda(\boldsymbol{x}) \in \mathbb{R}$,

$$\boldsymbol{a}\cdot\nabla f = f\;\boldsymbol{a}\cdot\nabla = \{\boldsymbol{a}\cdot\nabla\lambda(\boldsymbol{x})\}\frac{\partial g}{\partial \lambda}\,. \tag{2.15}$$

Proposition 2.7 (Left and right derivative from differential).

$$\nabla f = \nabla_{\boldsymbol{a}}\,(\boldsymbol{a}\cdot\nabla f), \quad \dot{f}\dot{\nabla} = (\dot{\boldsymbol{a}}\cdot\nabla f)\dot{\nabla}_{\boldsymbol{a}}\,. \tag{2.16}$$

Proposition 2.8 (Left and right product rules).

$$\nabla(fg) = (\dot{\nabla}\dot{f})g + \dot{\nabla}f\dot{g} = (\dot{\nabla}\dot{f})g + \nabla_{\boldsymbol{a}}\,f(\boldsymbol{a}\cdot\nabla g)\,. \tag{2.17}$$

$$(fg)\nabla = f(\dot{g}\dot{\nabla}) + \dot{f}g\dot{\nabla} = f(\dot{g}\dot{\nabla}) + (\dot{\boldsymbol{a}}\cdot\nabla f)\,g\,\dot{\nabla}_{\boldsymbol{a}}\,. \tag{2.18}$$

Note that the multivector functions f and g in (2.17) and (2.18) do not necessarily commute.

Differentiating twice with the vector derivative, we get the differential Laplace operator ∇^2. We can always write $\nabla^2 = \nabla\cdot\nabla + \nabla\wedge\nabla$, but for integrable functions one has that $\nabla\wedge\nabla = 0$, which then leads to $\nabla^2 = \nabla\cdot\nabla$. As ∇^2 is a scalar operator, the left and right Laplace derivatives agree, i.e. $\nabla^2 f = f\,\nabla^2$. More generally, all even powers of the left and right vector derivative agree:

$$\nabla^m f = f\,\nabla^m \quad \text{for} \quad m = 2m' \;(m' \in \mathbb{N})\,. \tag{2.19}$$

§3. Clifford analysis

The function theory for the operator ∇, often denoted by means of $\underline{\partial}_x$ in the literature (and referred to as the Dirac operator), is known as Clifford analysis. This is a multivariate function theory, which can be described as a higher-dimensional version of complex analysis or a refinement of harmonic analysis on $\mathbb{R}^n$. The latter is a simple consequence of the fact that ∇^2 yields the Laplace operator, the former refers to the fact that the functions on which ∇ acts take their values in the geometric algebra $\mathcal{G}_n$, which then generalizes the algebra $\mathbb{C}$ of complex numbers.

Remark 3.1. Note that in classical Clifford analysis, for which we refer to e.g. [2, 11, 14, 15], one usually works with the geometric algebra (also known as a Clifford algebra) of signature $(0, n)$, which lies closer to the idea of having complex units (n non-commuting complex units $e_j^2 = -1$, to be precise). However, in this paper we have chosen to work with the geometric algebra $\mathcal{G}_n$ associated to the Euclidean signature $(n, 0)$ to stay closer to the situation as it is used in physics. It is important to add that this has little influence on the final conclusions, as most results in Clifford analysis (especially the ones we need in this paper) can be formulated independent of the signature.

Clifford analysis can then essentially be described as the function theory for the Dirac operator, in which properties of functions $f(\boldsymbol{x}) \in \ker \nabla$ are studied. In this section, we will list a few properties. For the main part, we refer to the aforementioned references, or the excellent overview paper [10]. An important definition, in which the analogues of holomorphic powers z^k are introduced, is the following:

Definition 3.2. For all integers $k \geq 0$, the vector space $\mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n)$ is defined by means of

$$\mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n) := \mathrm{Pol}_k(\mathbb{R}^n, \mathcal{G}_n) \cap \ker \nabla \ . \tag{3.1}$$

This is the vector space of k-homogeneous monogenics on $\mathbb{R}^n$, containing polynomial null solutions for the Dirac operator.

As the Dirac operator ∇ is surjective on polynomials, see e.g. [11], one easily finds for all $k \geq 1$ that

$$\begin{aligned} d_k &:= \dim \mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n) \\ &= \dim \mathcal{P}_k(\mathbb{R}^n, \mathcal{G}_n) - \dim \mathcal{P}_{k-1}(\mathbb{R}^n, \mathcal{G}_n) \\ &= 2^n \binom{n+k-2}{k} \ . \end{aligned} \tag{3.2}$$

It is trivial to see that $d_0 = \dim \mathcal{G}_n$. In view of the fact that the operator ∇ factorizes the Laplace operator, null solutions of which are called harmonics, it is obvious that each monogenic polynomial is also harmonic. Indeed, denoting the space of k-homogeneous ($\mathcal{G}_n$-valued) harmonic polynomials by means of $\mathcal{H}_k(\mathbb{R}^n, \mathcal{G}_n)$, one has for all $k \in \mathbb{N}$ that

$$\mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n) \subset \mathcal{H}_k(\mathbb{R}^n, \mathcal{G}_n) \ . \tag{3.3}$$

As ∇^2 is a scalar operator, one can even decompose monogenic polynomials into harmonic ones (using the basis for $\mathcal{G}_n$):

$$M_k(\boldsymbol{x}) = \sum_A \boldsymbol{e}_A M_k^{(A)}(\boldsymbol{x}) \Rightarrow \nabla^2 M_k^{(A)}(\boldsymbol{x}) = 0 \ . \tag{3.4}$$

This will turn out to be an important property, since we will use a crucial algebraic characterization of the space of real-valued harmonic polynomials in the next section. Another crucial decomposition, which will be used in the next section, is the so-called Fischer decomposition for $\mathcal{G}_n$-valued polynomials on $\mathbb{R}^n$ (see e.g. [11] for a proof):

Theorem 3.3. *For any $k \in \mathbb{N}$, the space $\mathcal{P}_k(\mathbb{R}^n, \mathcal{G}_n)$ decomposes into a direct sum of monogenic polynomials:*

$$\mathcal{P}_k(\mathbb{R}^n, \mathcal{G}_n) = \bigoplus_{j=0}^{k} \boldsymbol{x}^j \mathcal{M}_{k-j}(\mathbb{R}^n, \mathcal{G}_n) \ . \tag{3.5}$$

We will often rely on the fact that the operator ∇ and the one-vector $\boldsymbol{x} \in \mathcal{G}_n$ (considered as a multiplication operator, acting on $\mathcal{G}_n$-valued functions) span the Lie superalgebra $\mathfrak{osp}(1,2)$, which appears as a Howe dual partner for the spin group (see [3] for more details). This translates itself into a collection of operator identities, and we hereby list the most crucial ones for what follows. Note that $\mathbb{E}_x := \sum_j x_j \partial_{x_j}$ denotes the Euler operator on $\mathbb{R}^m$, and $\{A, B\} = AB + BA$ (resp. $[A, B] = AB - BA$) denotes the anti-commutator (resp. the commutator) of two operators A and B:

$$\begin{aligned}
\{\boldsymbol{x}, \nabla\} &= 2\left(\mathbb{E}_x + \tfrac{n}{2}\right) & [\nabla^2, ||\boldsymbol{x}||^2] &= 4\left(\mathbb{E}_x + \tfrac{n}{2}\right) \\
\{\boldsymbol{x}, \boldsymbol{x}\} &= 2||\boldsymbol{x}||^2 & [\nabla, ||\boldsymbol{x}||^2] &= 2\boldsymbol{x} \\
\{\nabla, \nabla\} &= 2\nabla^2 & [\nabla^2, \boldsymbol{x}] &= 2\nabla
\end{aligned}$$

In particular, there also exists an operator which anti-commutes with the generators $\boldsymbol{x}$ and ∇ of $\mathfrak{osp}(1,2)$, see [1]. This operator, which is known as the Scasimir operator in abstract representation theory (it factorizes the Casimir operator), is related to the Gamma operator from Clifford analysis. To define this latter operator, we need a polar decomposition of the Dirac operator. Due to the change of signature, mentioned in the remark above, we will find a Gamma operator which differs from the one obtained in e.g. [11] by an overall minus sign. In order to retain the most important properties of this operator, we will therefore introduce a new notation Γ_∇, for the operator defined below[4]:

$$\boldsymbol{x}\nabla = \mathbb{E}_x + \Gamma_\nabla := \sum_{j=1}^{n} x_j\partial_{x_j} + \sum_{i<j} e_{ij}(x_i\partial_{x_j} - x_j\partial_{x_i}) \ . \tag{3.6}$$

Remark 3.4. Note that the operators $dH^x_{ij} := x_i\partial_{x_j} - x_j\partial_{x_i}$ appearing in the definition for the Gamma operator are known as the angular momentum operators. For $n = 3$, they are denoted by means of (L_x, L_y, L_z) and appear in quantum mechanics as the generators of the Lie algebra $\mathfrak{so}(3)$. In the next section, we will need the higher-dimensional analogue.

Definition 3.5. The Scasimir operator is defined by means of

$$Sc := \frac{1}{2}[\boldsymbol{x}, \nabla] + \frac{1}{2} = \Gamma_\nabla - \frac{n-1}{2} \ . \tag{3.7}$$

This operator satisfies

$$\{Sc, \boldsymbol{x}\} = 0 = \{Sc, \nabla\} = 0 \ , \tag{3.8}$$

which can easily be verified (although this is superfluous, as the operator is specifically designed to satisfy these two conditions, see [1]). For example, one has that

$$\{Sc, \boldsymbol{x}\} = \frac{1}{2}\{[\boldsymbol{x}, \nabla] + 1, \boldsymbol{x}\} = \frac{1}{2}[\boldsymbol{x}, \nabla]\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}[\boldsymbol{x}, \nabla] + \boldsymbol{x} = 0 \ ,$$

hereby using that $[\boldsymbol{x}^2, \nabla] = -2\boldsymbol{x}$ (see the relations above). Note also that the spaces $\mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n)$ are eigenspaces for the Gamma operator Γ_∇, which follows from the polar decomposition (3.6):

$$\Gamma_\nabla\big(M_k(\boldsymbol{x})\big) = -kM_k(\boldsymbol{x}) \qquad \big(\forall M_k(\boldsymbol{x}) \in \mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n)\big) \ . \tag{3.9}$$

As a result, one also has that

$$Sc\big(M_k(\boldsymbol{x})\big) = -\left(k + \frac{n-1}{2}\right) M_k(\boldsymbol{x}) \qquad \big(\forall M_k(\boldsymbol{x}) \in \mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n)\big) \ . \tag{3.10}$$

In the next section, we will also need the following:

Lemma 3.6. *If $f(r) = f(||\boldsymbol{x}||)$ denotes a scalar radial function on $\mathbb{R}^n$, one has for all $M_k(\boldsymbol{x}) \in \mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n)$ that*

$$Sc\big[M_k(\boldsymbol{x})f(r)\big] = -\left(k + \frac{n-1}{2}\right) M_k(\boldsymbol{x})f(r) \ .$$

[4] One simply has that $\Gamma_\nabla = -\Gamma$.

Proof: this follows from the fact that

$$\begin{aligned} Sc\big[M_k(\boldsymbol{x})f(r)\big] &= \left(\Gamma_\nabla - \frac{n-1}{2}\right)\big[M_k(\boldsymbol{x})f(r)\big] \\ &= [\Gamma_\nabla M_k(\boldsymbol{x})]f(r) - \frac{n-1}{2}M_k(\boldsymbol{x})f(r) \\ &= -\left(k + \frac{n-1}{2}\right)M_k(\boldsymbol{x})f(r) \ . \end{aligned}$$

We hereby used the fact that $\Gamma_\nabla[f(r)] = 0$, which follows from $dH^x_{ij}(r) = 0$ for all $i < j$. □

§4. The Clifford Fourier Transform

Let us recall the following definition, for which we refer to [19]:

Definition 4.1 (Clifford Fourier Transform).
The Clifford Fourier Transform (CFT) of a function $f(\boldsymbol{x}) : \mathbb{R}^n \to \mathcal{G}_n$, with $n = 2, 3 \pmod 4$, is the function

$$\begin{aligned} &\mathcal{F}\{f\} : \mathbb{R}^n \to \mathcal{G}_n \\ &\boldsymbol{\omega} \mapsto \mathcal{F}\{f\}(\boldsymbol{\omega}) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(\boldsymbol{x})e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}d^n\boldsymbol{x} \ , \end{aligned} \tag{4.1}$$

where $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^n$ and $d^n\boldsymbol{x} = (dx_1\boldsymbol{e_1} \wedge dx_2\boldsymbol{e_2} \wedge \cdots \wedge dx_n\boldsymbol{e_n})i_n^{-1}$. Also note that in the formula above we have added the factor $(2\pi)^{-\frac{n}{2}}$, which will simplify our expression for the Gaussian eigenfunction (see below).

For a complete list of the properties of this integral transform, which acts on $\mathcal{G}_n$-valued functions, we refer to Table 1 (see [19]). However, in the present paper, the following properties will play a crucial role:

Proposition 4.2.
For functions $f(\boldsymbol{x}) : \mathbb{R}^n \to \mathcal{G}_n$, the CFT satisfies the following:

(i) powers of $\boldsymbol{x}$ from the left:

$$\mathcal{F}\{\boldsymbol{x}^m f\}(\boldsymbol{\omega}) = \nabla^m_{\boldsymbol{\omega}}\mathcal{F}\{f\}(\boldsymbol{\omega})i_n^m \ , \quad m \in \mathbb{N} \ .$$

(ii) powers of $\boldsymbol{a}\cdot\boldsymbol{x}$:

$$\mathcal{F}\{(\boldsymbol{a}\cdot\boldsymbol{x})^m f\}(\boldsymbol{\omega}) = (\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}})^m\mathcal{F}\{f\}(\boldsymbol{\omega})i_n^m \ , \quad m \in \mathbb{N} \ .$$

(iii) vector derivatives from the left:

$$\mathcal{F}\{\nabla^m_{\boldsymbol{x}} f\}(\boldsymbol{\omega}) = \boldsymbol{\omega}^m\mathcal{F}\{f\}(\boldsymbol{\omega})i_n^m \ , \quad m \in \mathbb{N} \ .$$

Table 1. Properties of the Clifford Fourier transform (CFT) of Definition 4.1 with $n = 2, 3 \pmod 4$. Multivector functions (Multiv. Funct.) f, g, f_1, f_2 all belong to $L^2(\mathbb{R}^n, \mathcal{G}_n)$, the constants are $\alpha, \beta \in \mathcal{G}_n$, $0 \neq a \in \mathbb{R}$, $\boldsymbol{a}, \boldsymbol{\omega}_0 \in \mathbb{R}^n$ and $m \in \mathbb{N}$.

Property	**Multiv. Funct.**	**CFT**
Left lin.	$\alpha f(\boldsymbol{x}) + \beta\, g(\boldsymbol{x})$	$\alpha \mathcal{F}\{f\}(\boldsymbol{\omega}) + \beta \mathcal{F}\{g\}(\boldsymbol{\omega})$
$\boldsymbol{x}$-Shift	$f(\boldsymbol{x} - \boldsymbol{a})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{a}}$
$\boldsymbol{\omega}$-Shift	$f(\boldsymbol{x})\, e^{i_n \boldsymbol{\omega}_0\cdot\boldsymbol{x}}$	$\mathcal{F}\{f\}(\boldsymbol{\omega} - \boldsymbol{\omega}_0)$
Scaling	$f(a\boldsymbol{x})$	$\frac{1}{\lvert a\rvert^n}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$
Vec. diff.	$(\boldsymbol{a}\cdot\nabla)^m f(\boldsymbol{x})$	$(\boldsymbol{a}\cdot\boldsymbol{\omega})^m \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
	$(\boldsymbol{a}\cdot\boldsymbol{x})^m\, f(\boldsymbol{x})$	$(\boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}})^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
Powers of $\boldsymbol{x}$	$\boldsymbol{x}^m f(\boldsymbol{x})$	$\nabla_{\boldsymbol{\omega}}^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$\boldsymbol{\omega}^m\, \mathcal{F}\{f\}(\boldsymbol{\omega})\, i_n^m$
Plancherel	$\int_{\mathbb{R}^n} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}\, d^n\boldsymbol{x}$	$\int_{\mathbb{R}^n} \mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\, d^n\boldsymbol{\omega}$
sc. Parseval	$\int_{\mathbb{R}^n} \lVert f(\boldsymbol{x})\rVert^2\, d^n\boldsymbol{x}$	$\int_{\mathbb{R}^n} \lVert\mathcal{F}\{f\}(\boldsymbol{\omega})\rVert^2\, d^n\boldsymbol{\omega}$

(iv) directional derivatives:

$$\mathcal{F}\{(\boldsymbol{a}\cdot\nabla_{\boldsymbol{x}})^m f\}(\boldsymbol{\omega}) = (\boldsymbol{a}\cdot\boldsymbol{\omega})^m \mathcal{F}\{f\}(\boldsymbol{\omega}) i_n^m \ , \quad m \in \mathbb{N} \ .$$

In order to obtain an operator exponential expression for the CFT from above, see definition 4.1, we need to construct a family of eigenfunctions which then moreover serves as a basis for the function space $\mathcal{S}(\mathbb{R}^n, \mathcal{G}_n)$ of rapidly decreasing test functions taking values in $\mathcal{G}_n$. To do so, we need a series of results:

Proposition 4.3.
The Gaussian function $G(\boldsymbol{x}) := \exp(-\frac{1}{2}\boldsymbol{x}^2) = \exp(-\frac{1}{2}||\boldsymbol{x}||^2)$ on $\mathbb{R}^n$ defines an eigenfunction for the CFT:

$$\mathcal{F}\{G(\boldsymbol{x})\}(\boldsymbol{\omega}) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp(-\frac{1}{2}\boldsymbol{x}^2) e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{x} = G(\boldsymbol{\omega}) \ .$$

Proof: this follows from a similar property in [19], taking the rescaling factor $(2\pi)^{-\frac{n}{2}}$ into account. □

Next, we prove that the Gaussian $G(\boldsymbol{x})$ may be multiplied with arbitrary monogenic polynomials: this will still yield an eigenfunction for $\mathcal{F}$. In order to prove this, we recall formula (3.4):

$$M_k(\boldsymbol{x}) \in \mathcal{P}_k(\mathbb{R}^n, \mathcal{G}_n) \cap \ker \nabla \implies M_k(\boldsymbol{x}) = \sum_A \boldsymbol{e}_A M_k^{(A)}(\boldsymbol{x}) \ , \tag{4.2}$$

with each $M_k^{(A)}(\boldsymbol{x}) \in \mathcal{H}_k(\mathbb{R}^n, \mathbb{R})$ a real-valued harmonic polynomial on $\mathbb{R}^n$. This allows us to focus our attention on *harmonic* polynomials. As is well-known, the vector space $\mathcal{H}_k(\mathbb{R}^n, \mathbb{C})$ of k-homogeneous harmonic polynomials in n real variables defines an irreducible module for the special orthogonal group $\mathrm{SO}(n)$ or its Lie algebra $\mathfrak{so}(n)$. This algebra is spanned by the $\binom{n}{2}$ angular momentum operators

$$dH_{ij}^x := x_i(\boldsymbol{e_j} \cdot \nabla) - x_j(\boldsymbol{e_i} \cdot \nabla) = x_i\partial_{x_j} - x_j\partial_{x_i} \quad (1 \leq i < j \leq n), \tag{4.3}$$

see remark 3.4. It then follows from general Lie theoretical considerations that the vector space $\mathcal{H}_k(\mathbb{R}^n, \mathbb{C})$ is generated by the repeated action of the negative root vectors in $\mathfrak{so}(m)$ acting on a unique highest weight vector. For the representation space $\mathcal{H}_k(\mathbb{R}^n, \mathbb{C})$, this highest weight vector is given by $h_k(\boldsymbol{x}) := (x_1 - ix_2)^k$, see e.g. [5, 14]. Without going into too much detail, it suffices to understand that this implies that arbitrary elements in $\mathcal{H}_k(\mathbb{R}^n, \mathbb{C})$ can always be written as

$$H_k(\boldsymbol{x}) = \mathcal{L}(dH_{ij}^x)h_k(\boldsymbol{x}) \ ,$$

where $\mathcal{L}(dH_{ij}^x)$ denotes some linear combination[5] of products of the angular momentum operators dH_{ij}^x. Note that the presence of the complex number field in the argument above has no influence on the fact that we are working with functions taking values in the *real* algebra $\mathcal{G}_n$ in this paper: it suffices to focus on the real (or pure imaginary) part afterwards.

Theorem 4.4.
Given an arbitrary element $M_k(\boldsymbol{x}) \in \mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n)$, one has:

$$\mathcal{F}\{M_k(\boldsymbol{x})G(\boldsymbol{x})\}(\boldsymbol{\omega}) = M_k(\boldsymbol{\omega})G(\boldsymbol{\omega})(-i_n)^k \ .$$

Proof: in view of the decomposition (4.2), we have that

$$\mathcal{F}\{M_k(\boldsymbol{x})G(\boldsymbol{x})\}(\boldsymbol{\omega}) = \sum_A e_A \mathcal{F}\{M_k^{(A)}(\boldsymbol{x})G(\boldsymbol{x})\}(\boldsymbol{\omega}) \ .$$

If we then write each scalar component $M_k^{(A)}(\boldsymbol{x})$ as a linear combination of the form

$$M_k^{(A)}(\boldsymbol{x}) = \mathcal{L}^{(A)}(dH_{ij}^x)h_k(\boldsymbol{x}) \ ,$$

we are clearly left with the analysis of terms of the following type (with $1 \leq i < j \leq n$ arbitrary):

$$\mathcal{F}\{\mathcal{L}^{(A)}(dH_{ij}^x)(x_1 - ix_2)^k G(\boldsymbol{x})\}(\boldsymbol{\omega}) \ .$$

[5] In a sense, each combination $\mathcal{L}(dH_{ij}^x)$ can be seen as some sort of non-commutative polynomial (the factors dH_{ij}^x do not commute).

Without loosing generality, we can focus our attention on a single operator dH_{ij}^x, since any $\mathcal{L}^{(A)}(dH_{ij}^x)$ can always be written as a sum of products of these operators. Invoking properties from proposition 4.2, it is clear that

$$\begin{aligned}\mathcal{F}\{dH_{ij}^x h_k G\}(\boldsymbol{\omega}) &= \mathcal{F}\big\{(x_i(\boldsymbol{e_j}\cdot\nabla_{\boldsymbol{x}}) - x_j(\boldsymbol{e_i}\cdot\nabla_{\boldsymbol{x}}))h_k G\big\}(\boldsymbol{\omega})\\ &= (\omega_j(\boldsymbol{e_i}\cdot\nabla_{\boldsymbol{\omega}}) - \omega_i(\boldsymbol{e_j}\cdot\nabla_{\boldsymbol{\omega}}))\mathcal{F}\big\{h_k G\big\}(\boldsymbol{\omega})i_n^2\\ &= dH_{ij}^\omega \mathcal{F}\big\{h_k G\big\}(\boldsymbol{\omega})\ ,\end{aligned}$$

where dH_{ij}^ω denotes the angular momentum operator in the variable $\boldsymbol{\omega}\in\mathbb{R}^n$. Next, invoking the same proposition, we also have that

$$\begin{aligned}\mathcal{F}\big\{h_k G\big\}(\boldsymbol{\omega}) &= \mathcal{F}\big\{(x_1 - ix_2)^k G\big\}(\boldsymbol{\omega})\\ &= \big((\boldsymbol{e_1}\cdot\nabla_{\boldsymbol{\omega}}) - i(\boldsymbol{e_2}\cdot\nabla_{\boldsymbol{\omega}})\big)^k \mathcal{F}\{G\}(\boldsymbol{\omega})i_n^k\\ &= \big((\boldsymbol{e_1}\cdot\nabla_{\boldsymbol{\omega}}) - i(\boldsymbol{e_2}\cdot\nabla_{\boldsymbol{\omega}})\big)^k G(\boldsymbol{\omega})i_n^k\ .\end{aligned}$$

It then suffices to note that

$$\big((\boldsymbol{e_1}\cdot\nabla_{\boldsymbol{\omega}}) - i(\boldsymbol{e_2}\cdot\nabla_{\boldsymbol{\omega}})\big)^k \exp(-\frac{1}{2}||\boldsymbol{\omega}||^2) = (-\omega_1 + i\omega_2)^k \exp(-\frac{1}{2}||\boldsymbol{\omega}||^2)$$

to arrive at $\mathcal{F}\big\{h_k G\big\}(\boldsymbol{\omega}) = (-1)^k h_k(\boldsymbol{\omega})G(\boldsymbol{\omega})i_n^k$. Together with what was found above, this then proves the theorem. □

Remark 4.5. There exist alternative ways to prove the fact that harmonic polynomials times a Gaussian kernel define eigenfunctions for the Fourier transform, see e.g. the seminal work [23] by Stein and Weiss, but to our best knowledge the proof above has not appeared in the literature yet.

In order to arrive at a basis of eigenfunctions for the space $\mathcal{S}(\mathbb{R}^n, \mathcal{G}_n)$, we need more eigenfunctions for the CFT. For that purpose, we introduce the following definition (recall that we have defined d_k in the previous section).

Definition 4.6.
For a given monogenic polynomial $M_k^{(b)}(\boldsymbol{x}) \in \mathcal{M}_k(\mathbb{R}^n, \mathcal{G}_n)$, where the index $b \in \{1, 2, \ldots, d_k\}$ is used to label a basis for the vector space of monogenics of degree k, we define the Clifford-Hermite eigenfunctions as

$$\varphi_{a,b;k}(\boldsymbol{x}) := (\nabla - \boldsymbol{x})^a M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x})\ .$$

Hereby, the index $a \geq 0$ denotes an arbitrary non-negative integer.

Theorem 4.7.
For all indices $(a, b; k) \in \mathbb{N}\times\{1, \ldots, d_k\}\times\mathbb{N}$, *one has that*

$$\mathcal{F}\{\varphi_{a,b;k}\}(\boldsymbol{\omega}) = \varphi_{a,b;k}(\boldsymbol{\omega})(-i_n)^{a+k}\ .$$

Proof: this can again be proved using proposition 4.2. Indeed, we clearly have that

$$\begin{aligned}\mathcal{F}\{\varphi_{a,b;k}\}(\boldsymbol{\omega}) &= \mathcal{F}\{(\nabla_{\boldsymbol{x}} - \boldsymbol{x})^a M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x})\}(\boldsymbol{\omega})\\ &= (\boldsymbol{\omega} - \nabla_{\boldsymbol{\omega}})^a \mathcal{F}\{M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x})\}(\boldsymbol{\omega}) i_n^a\\ &= (\nabla_{\boldsymbol{\omega}} - \boldsymbol{\omega})^a M_k^{(b)}(\boldsymbol{\omega})G(\boldsymbol{\omega})(-i_n)^k(-i_n)^a\\ &= \varphi_{a,b;k}(\boldsymbol{\omega})(-i_n)^{a+k} \ ,\end{aligned}$$

where we have made use of theorem 4.4. □

Before we come to an exponential operator form for the CFT, we prove a few additional results:

Lemma 4.8.
For all indices $(a, b; k) \in \mathbb{N} \times \{1, \ldots, d_k\} \times \mathbb{N}$*, one has:*

$$(\nabla + \boldsymbol{x})\varphi_{2a,b;k}(\boldsymbol{x}) = -4a\varphi_{2a-1,b;k}(\boldsymbol{x})$$

$$(\nabla + \boldsymbol{x})\varphi_{2a+1,b;k}(\boldsymbol{x}) = -2(n + 2k + 2a)\varphi_{2a,b;k}(\boldsymbol{x}) \ .$$

Proof: this lemma can easily be proved by induction on the parameter $a \in \mathbb{N}$, hereby taking into account that

$$[\nabla + \boldsymbol{x}, \nabla - \boldsymbol{x}] = 2[\boldsymbol{x}, \nabla] = 4Sc - 2 \ ,$$

with $Sc \in \mathfrak{osp}(1, 2)$ the Scasimir operator defined in (3.7). Indeed, for $a = 0$ we immediately get that

$$\begin{aligned}(\nabla + \boldsymbol{x})\varphi_{0,b;k} &= \dot{\nabla} G(\dot{\boldsymbol{x}}) M_k^{(b)}(\boldsymbol{x}) + \boldsymbol{x}\varphi_{0,b;k}\\ &= -\boldsymbol{x}G(\boldsymbol{x})M_k^{(b)}(\boldsymbol{x}) + \boldsymbol{x}\varphi_{0,b;k} = 0 \ .\end{aligned}$$

For $a = 1$, we get that

$$\begin{aligned}(\nabla + \boldsymbol{x})\varphi_{1,b;k} &= \big((\nabla - \boldsymbol{x})(\nabla + \boldsymbol{x}) + 2(2Sc - 1)\big)\varphi_{0,b;k}\\ &= -2(n + 2k)\varphi_{0,b;k} \ .\end{aligned}$$

Here we have used the fact (3.10) that monogenic homogeneous polynomials are eigenfunctions for the Scasimir operator, together with lemma 3.6. Let us then for example consider a general odd index $2a + 1$ (the case of an even index $2a$ is completely similar):

$$\begin{aligned}(\nabla + \boldsymbol{x})\varphi_{2a+1,b;k} &= \big((\nabla - \boldsymbol{x})(\nabla + \boldsymbol{x}) + 2(2Sc - 1)\big)\varphi_{2a,b;k}\\ &= -4a(\nabla - \boldsymbol{x})\varphi_{2a-1,b;k} + 2(\nabla - \boldsymbol{x})^{2a}(2Sc - 1)\varphi_{0,b;k}\\ &= -2(2a + n + 2k)\varphi_{2a,b;k} \ .\end{aligned}$$

Here, we have used the induction hypothesis and the fact that the operator Sc commutes with an even power $(\nabla - \boldsymbol{x})$, as it anti-commutes with each individual factor. □

This lemma will now be used to construct another operator, for which our Clifford-Hermite functions from definition 4.6 are again eigenfunctions.

Theorem 4.9.
For all indices $(a, b; k) \in \mathbb{N} \times \{1, \ldots, d_k\} \times \mathbb{N}$*, one has:*

$$(\nabla^2 - \boldsymbol{x}^2)\varphi_{a,b;k}(\boldsymbol{x}) = -(n + 2a + 2k)\varphi_{a,b;k}(\boldsymbol{x}) \ .$$

Proof: first of all, we note that

$$(\nabla + \boldsymbol{x})(\nabla - \boldsymbol{x}) = \nabla^2 - \boldsymbol{x}^2 + [\boldsymbol{x}, \nabla] \ ,$$

from which we note that the operator appearing in the theorem can also be written as

$$\nabla^2 - \boldsymbol{x}^2 = (\nabla + \boldsymbol{x})(\nabla - \boldsymbol{x}) - (2Sc - 1) \ .$$

Using lemma 4.8 and properties of the Scasimir operator, we get:

$$\begin{aligned}
(\nabla^2 - \boldsymbol{x}^2)\varphi_{2a,b;k} &= \big((\nabla + \boldsymbol{x})(\nabla - \boldsymbol{x}) - (2Sc - 1)\big)\varphi_{2a,b;k} \\
&= (\nabla + \boldsymbol{x})\varphi_{2a+1,b;k} - (\nabla - \boldsymbol{x})^{2a}(2Sc - 1)\varphi_{0,b;k} \\
&= -2(n + 2k + 2a)\varphi_{2a,b;k} + (n + 2k)\varphi_{2a,b;k} \\
&= -(n + 2k + 4a)\varphi_{2a,b;k}
\end{aligned}$$

for the case of an even index $2a$, and

$$\begin{aligned}
(\nabla^2 - \boldsymbol{x}^2)\varphi_{2a+1,b;k} &= \big((\nabla + \boldsymbol{x})(\nabla - \boldsymbol{x}) - (2Sc - 1)\big)\varphi_{2a+1,b;k} \\
&= (\nabla + \boldsymbol{x})\varphi_{2a+2,b;k} + (\nabla - \boldsymbol{x})^{2a+1}(2Sc + 1)\varphi_{0,b;k} \\
&= -2(2a + 2)\varphi_{2a+1,b;k} - (n + 2k - 2)\varphi_{2a+1,b;k} \\
&= -(n + 2k + 4a + 2)\varphi_{2a+1,b;k}
\end{aligned}$$

for odd indices $2a + 1$. Together, this proves the theorem. □

In order to compare the eigenvalues of the Clifford-Hermite functions as eigenfunctions for the CFT and the operator from the theorem above (which is nothing but the Hamiltonian of the harmonic oscillator), we mention the following remarkable property:

$$\begin{aligned}\varphi_{a,b;k}(\dot{\boldsymbol{x}})e^{-\frac{\pi}{4}(\boldsymbol{x}^2-\dot{\nabla}^2-n)i_n} &= \sum_{j=0}^{\infty}\frac{1}{j!}(\boldsymbol{x}^2-\nabla^2-n)^j\varphi_{a,b;k}(\boldsymbol{x})\left(-\frac{\pi}{4}i_n\right)^j \\ &= \varphi_{a,b;k}(\boldsymbol{x})\sum_{j=0}^{\infty}\frac{1}{j!}\left(-(a+k)\frac{\pi}{2}i_n\right)^j \\ &= \varphi_{a,b;k}(\boldsymbol{x})(-i_n)^{a+k}\ . \end{aligned} \tag{4.4}$$

In the first line above, we have used the Hestenes' overdot notation, to stress the fact that the operator acts on $\boldsymbol{x}$ from the right. This is due to the fact that the pseudoscalar i_n does *not* necessarily commute with the Clifford-Hermite eigenfunction.

It thus suffices to prove that these eigenfunctions provide a basis for the function space $\mathcal{S}(\mathbb{R}^n,\mathcal{G}_n)$ in order to arrive at the main result of our paper, which is the exponential operator form for the CFT.

Proposition 4.10.
The space $\mathcal{S}(\mathbb{R}^n,\mathcal{G}_n)$ is spanned by the countable basis

$$\mathcal{B} := \left\{\varphi_{a,b;k}(\boldsymbol{x}) : (a,b;k)\in\mathbb{N}\times\{1,\ldots,d_k\}\times\mathbb{N}\right\}\ .$$

Proof: in order to prove this, it suffices to show that we can express arbitrary elements of $\mathcal{P}(\mathbb{R}^n,\mathcal{G}_n)\otimes G(\boldsymbol{x})$ as a linear combination of the Clifford-Hermite eigenfunctions. To do so, we can use the fact that we know the structure of the space of $\mathcal{G}_n$-valued polynomials in terms of the Fischer decomposition, see (3.5). It thus suffices to note that

$$(\nabla-\boldsymbol{x})^a M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x}) = (-2)^a\boldsymbol{x}^a M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x}) + \text{L.O.T.}\ ,$$

where L.O.T. refers to lower powers in $\boldsymbol{x}$ times the product of $M_k^{(b)}(\boldsymbol{x})$ and the Gaussian function. For $a\in\{0,1\}$, this is trivial, as we for example have that

$$\begin{aligned}(\nabla-\boldsymbol{x})M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x}) &= -\boldsymbol{x}M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x}) + \dot{\nabla}G(\dot{\boldsymbol{x}})M_k^{(b)}(\boldsymbol{x}) \\ &= -2\boldsymbol{x}M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x})\ ,\end{aligned}$$

and the rest follows from an easy induction argument. In case of an odd index $2a+1$ for example, the induction hypothesis gives:

$$(\nabla-\boldsymbol{x})^{2a+1}M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x}) = (\nabla-\boldsymbol{x})\left(4^a\boldsymbol{x}^{2a}M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x}) + \text{L.O.T.}\right)\ ,$$

where the first power in $\boldsymbol{x}$ appearing in L.O.T. is equal to $(2a-2)$. This is due to the fact that the operator $(\nabla-\boldsymbol{x})$ can only raise by one, either from the multiplication by $\boldsymbol{x}$ or the action of ∇ on $G(\boldsymbol{x})$, or lower by one, which comes from the action of ∇ on a power in $\boldsymbol{x}$. Using the relation

$$\nabla_{\boldsymbol{x}}\boldsymbol{x} = \boldsymbol{x}\nabla_{\boldsymbol{x}} - (2Sc - 1) \ ,$$

it is then easily seen that we indeed arrive at the constant $(-2)^{2a+1}$ for the leading term in $\boldsymbol{x}$. As all the leading terms are different, it follows that any function of the form $\boldsymbol{x}^a M_k^{(b)}(\boldsymbol{x})G(\boldsymbol{x})$ can indeed be expressed as a unique linear combination of the Clifford-Hermite eigenfunctions. □

Bringing everything together, we have thus obtained from theorem 4.7, equation (4.4), and proposition 4.10, the following final result:

Theorem 4.11 (CFT as exponential operator). *The Clifford Fourier transform, as an operator on $\mathcal{S}(\mathbb{R}^n, \mathcal{G}_n)$, can be defined by means of*

$$\begin{aligned} \mathcal{F}\{f\} &: \mathbb{R}^n \to \mathcal{G}_n \\ &\boldsymbol{\omega} \mapsto \mathcal{F}\{f\}(\boldsymbol{\omega}) = f(\dot{\boldsymbol{\omega}}) e^{-\frac{\pi}{4}(\boldsymbol{\omega}^2 - \dot{\nabla}_{\boldsymbol{\omega}}^2 - n) i_n} \ . \end{aligned}$$

In other words, the CFT can be written as an exponential operator acting from the right, involving the Hamiltonian $\boldsymbol{\omega}^2 - \dot{\nabla}_{\boldsymbol{\omega}}^2$, for the harmonic oscillator, and a phase factor $e^{\frac{n\pi}{4} i_n}$ which reduces to $(i_3 - 1)/\sqrt{2}$ for $n = 3$ and to i_2 for $n = 2$.

§5. Conclusion

The Clifford Fourier transform originated as a tool for electro-magnetic field processing, applying Clifford's geometric algebra and Clifford analysis to physics. The Fourier transform itself being one of the most widely applied transformations in mathematics. The harmonic oscillator and its Hamilton operator are fundamental for the description of periodic harmonic motions both in classical and quantum physics. In the latter the Fourier transform yields the momentum representation and explains the uncertainty principle for fields (signals) and their Fourier transformed counterparts. In this paper we found, that the multivector integral of the Clifford Fourier transform has an equivalent form as the exponential of the Hamilton operator of a harmonic oscillator. This interesting observation adds a fascinating new facet to the mathematics of multivector integral transformations and their wide ranging applications in physics and technology.

Possible future extensions are to multi-kernel CFTs, the quaternionic CFT and its higher-dimensional analogues, like the spacetime Fourier transformation, Hamilton operators for other systems than the harmonic oscillator, and better understanding of the deep connections between seemingly separate ways [4] of generalizing the Fourier transform in Clifford's geometric algebra and Clifford analysis.

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