Nikodým-type theorems for lattice group-valued measures with respect to filter convergence

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Abstract

We present some convergence and boundedness theorem with respect to filter convergence for lattice group-valued measures, whose techniques are based on sliding hump arguments.

We give some new versions of Nikodým convergence, boundedness and Brooks-Jewett-type theorems with respect to filter convergence for lattice group-valued measures, defined on a σ-algebra of an abstract nonempty set, in which sliding hump-type techniques are used.

Let R be a Dedekind complete (ℓ)-group, Q be a countable set and $\mathcal{F}$ be a filter of Q. A subset of Q is *$\mathcal{F}$-stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of Q.

A filter $\mathcal{F}$ of Q is said to be *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$. Given an infinite set $I \subset Q$, a *blocking* of I is a countable partition $\{D_k : k \in \mathbb{N}\}$ of I into nonempty finite subsets.

A filter $\mathcal{F}$ of Q is said to be *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of I there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\sharp(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\sharp$ denotes the number of elements of the set into brackets.

If $I \in \mathcal{F}^*$, then the *trace* $\mathcal{F}(I)$ of $\mathcal{F}$ on I is the family $\{A \cap I : A \in \mathcal{F}\}$. It is not difficult to see that $\mathcal{F}(I)$ is a filter of I.

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Observe that, If $\mathcal{F}$ is a block-respecting filter of $\mathbb{N}$, then $\mathcal{F}(I)$ is a block-respecting filter of I for every $I \in \mathcal{F}^*$.

Let $\mathcal{F}$ be a filter of $\mathbb{N}$. A sequence $(x_n)_n$ in R $(D\mathcal{F})$-*converges to* $x \in R$ iff there is a (D)-sequence $(a_{t,l})_{t,l}$ with the property that $\left\{n \in \mathbb{N} : |x_n - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\right\} \in \mathcal{F}$ for each $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Let Ξ be any arbitrary nonempty set. A family $(\beta_{\xi,n})_{\xi\in\Xi,n\in\mathbb{N}}$ is said to be $(RD\mathcal{F})$-*convergent* to a family $(\beta_\xi)_{\xi\in\Xi}$ with respect to $\xi \in \Xi$ iff there is a regulator $(a_{t,l})_{t,l}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $\xi \in \Xi$ we get

$$\left\{n \in \mathbb{N} : |\beta_{\xi,n} - \beta_\xi| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\right\} \in \mathcal{F}.$$

Given $a < b \in R$, set $[a, b] = \{x \in R\colon\ a \le x \le b\}$. For A, $B \subset R$, $n \in \mathbb{N}$, put $A + B = \{a + b\colon\ a \in A,\ b \in B\}$, $n\,A = \{a + \ldots + a\}$ (n times). Let $U_n = [-u_n, u_n]$, $n \in \mathbb{N}$, be such that $0 < u_n \le u_{n+1}$ for every $n \in \mathbb{N}$. A set $\{x_n : n \in \mathbb{N}\} \subset R$ is said to be (PR)-$\mathcal{F}$-*bounded* by $(U_n)_n$, iff $\{n \in \mathbb{N} : x_n \in U_n\} \in \mathcal{F}$, and (PR)-*eventually bounded* by $(U_n)_n$ iff it is (PR)-$\mathcal{F}_{\text{cofin}}$-bounded by $(U_n)_n$.

We now give the main results.

Theorem 0.1. *Let R be a Dedekind complete (ℓ)-group, $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be a sequence of equibounded σ-additive measures, $(C_k)_k$ be a disjoint sequence in Σ, with*

(i) $(D)\lim_n m_n(C_k) = 0$ for any $k \in \mathbb{N}$, and

(ii) $(RD\mathcal{F})\lim_n m_n(\bigcup_{k\in P} C_k) = 0$ with respect to $P \in \mathcal{P}(\mathbb{N})$.

Then,

γ) for every strictly increasing sequence $(k_n)_n$ in $\mathbb{N}$ we get

$$(D\mathcal{F})\lim_n\, m_n(C_{k_n}) = 0; \tag{1}$$

$\gamma\gamma$) if $\mathcal{F}$ is also diagonal and R is super Dedekind complete and weakly σ-distributive, then the only condition (ii) is sufficient to get (1).

Theorem 0.2. *Let R be a Dedekind complete (ℓ)-group, $(C_k)_k$ be as in Theorem 0.1, $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be an equibounded sequence of finitely additive measures, and assume that*

(i) $(D)\lim_n m_n(C_k) = 0$ for any $k \in \mathbb{N}$;

(ii) $(RD\mathcal{F})\lim_n \sum_{k\in P} m_n(C_k) = 0$ with respect to $P \in \mathcal{P}(\mathbb{N})$.

Then for every strictly increasing sequence $(k_n)_n$ in $\mathbb{N}$ we get

$$(D\mathcal{F})\lim_n\, m_n(C_{k_n}) = 0. \tag{2}$$

If $\mathcal{F}$ is also diagonal and R is super Dedekind complete and weakly σ-distributive, then the only condition (ii) is enough to get (2).

Theorem 0.3. *Let R be any Dedekind complete (ℓ)-group, $u \in R$, $u > 0$, $U = [-u, u]$, $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of finitely additive measures, and assume that*

0.3.1) *for every disjoint sequence $(C_n)_n$ in Σ and $j \in \mathbb{N}$ there is $Q_j \subset \mathbb{N}$ with $\sum_{n \in Q} m_j(C_n) \in U$ for each $Q \subset Q_j$.*

Let $(C_n)_n$ be a disjoint sequence in Σ and $(w_n)_n$ be an increasing sequence of positive elements of R. For each $n \in \mathbb{N}$, set $W_n := [-w_n, w_n]$ and $V_n := n\,W_n + U$. Moreover suppose that:

(i) the set $\{m_n(C_p) : n \in \mathbb{N}\}$ is (PR)-eventually bounded by $(W_n)_n$ for each $p \in \mathbb{N}$;

(ii) the set $\left\{\sum_{p \in P} m_j(C_p) : n \in \mathbb{N}\right\}$ is (PR)-$\mathcal{F}$-bounded by $(W_n)_n$ for each $P \in \mathcal{P}(\mathbb{N})$.

Then we get:

(j) for every strictly increasing sequence $(l_n)_n$ in $\mathbb{N}$, the set $D := \{m_n(C_{l_n}) : n \in \mathbb{N}\}$ is (PR)-$\mathcal{F}$-bounded by $(V_n)_n$;

(jj) if $\mathcal{F}$ is also diagonal, then the only condition (ii) is enough in order that D is (PR)-$\mathcal{F}$-bounded by $(V_n)_n$.