The Minimal Length Stringy Uncertainty Relations follow from Clifford Space Relativity

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Abstract

We improve our earlier work in [14] and derive the minimal length string/membrane uncertainty relations by imposing momentum slices in flat Clifford spaces. The Jacobi identities associated with the modified Weyl-Heisenberg algebra require noncommuting spacetime coordinates, but commuting momenta, and which is compatible with the notion of curved momentum space. Clifford *Phase* Space Relativity requires the introduction of a *maximal* scale which can be identified with the Hubble scale and is a consequence of Born's Reciprocal Relativity Principle.

Keywords : Clifford algebras; Extended Relativity in Clifford Spaces; String Theory; Doubly Special Relativity; Noncommutative Geometry; Quantum Clifford-Hopf algebras.

1 Introduction : Novel Consequences of Clifford Space Relativity

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional

*Dedicated to the memory of Rachael Bowers target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1]. The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators, $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, $\{ \gamma_a, \gamma_b \} = 2 \; g_{ab} \; \mathbf{1}$, including the Clifford algebra unit element (associated to a scalar coordinate). These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background.

The C-space polyvector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions, for example,

$$\mathbf{X} \;=\; s \, \mathbf{1} \;+\; x^\mu \; \gamma_\mu \;+\; x^{\mu\nu} \; \gamma_\mu \wedge \gamma_\nu \;+\; x^{\mu\nu\rho} \; \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \;+\; x^{\mu\nu\rho\tau} \; \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\tau \quad (1.1)$$

where we have omitted combinatorial numerical factors for convenience in the expansion (1). It can be generalized to any dimensions, including $D = 0$. The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in C-space

$$(d\Sigma)^2 \;=\; (ds)^2 \;+\; dx_\mu \; dx^\mu \;+\; dx_{\mu\nu} \; dx^{\mu\nu} \;+\; \ldots \quad (1.2)$$

is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu} dx^\mu dx^\nu = dx_\mu dx^\mu$. In order to match units in all terms of eqs-(1.1,1.2) suitable powers of a length scale (say Planck scale) must be introduced. For convenience purposes it is can be set to unity. For extensive details of the generalized Lorentz transformations (poly-rotations) in flat C-spaces and references we refer to [1].

Let us now consider a basis in C-space given by

$$E_A = \gamma, \; \gamma_\mu, \; \gamma_\mu \wedge \gamma_\nu, \; \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, \; ... \quad (1.3)$$

where γ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In (3) when one writes an r-vector basis $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices in "lexicographical" order so that $\mu_1 < \mu_2 < .... < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X \;=\; X^A E_A \;=\; s \, \mathbf{1} \;+\; x^\mu \gamma_\mu \;+\; x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \quad (1.4)$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter L_p in the expansion of the polyvector X (in order to match units) we can set it to unity to simplify matters. In a *flat* C-space the basis vectors E^A, E_A are *constants*. In a *curved* C-space this is no longer true. Each E^A, E_A is a function of the C-space coordinates

$$X^A \;=\; \{\; s,\; x^\mu,\; x^{\mu_1\mu_2},\; .....,\; x^{\mu_1\mu_2.....\mu_D}\;\} \tag{1.5}$$

which include scalar, vector, bivector,..., p-vector,... coordinates in the underlying D-dim base spacetime and whose corresponding C-space is 2^D-dimensional since the Clifford algebra in D-dim is 2^D-dimensional.

The C-space metric is chosen to be $G^{AB} = 0$ when the *grade* $A \neq$ *grade* B. For the same-grade metric components $g^{[a_1a_2...a_k]\ [b_1b_2...b_k]}$ of G^{AB}, the metric can decomposed into its irreducible factors as antisymmetrized sums of products of η^{ab} given by the following *determinant* [13]

$$G^{AB} \;\equiv\; det \begin{pmatrix} \eta^{a_1b_1} & \ldots & \ldots & \eta^{a_1b_k} \\ \eta^{a_2b_1} & \ldots & \ldots & \eta^{a_2b_k} \\ ------------ & & --------------- & \\ \eta^{a_kb_1} & \ldots & \ldots & \eta^{a_kb_k} \end{pmatrix} \tag{1.6}$$

The spacetime signature is chosen to be $(-,+,+,....,+)$. One still has the freedom to choose the sign of the scalar-scalar components G_{**} of the C-space metric G_{AB}. In the next section we shall see that $G_{**} = -1 < 0$ is the right choice.

Recently, novel physical consequences of the Extended Relativity Theory in C-spaces (Clifford spaces) were explored in [4]. The latter theory provides a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework. Furthermore, an elegant *nonlinear* momentum-addition law was derived in order to tackle the "soccer-ball" problem in DSR. Neither derivation in C-spaces requires a *curved* momentum space nor a deformation of the Lorentz algebra. While the constant (energy-independent) speed of photon propagation is always compatible with the generalized photon dispersion relations in C-spaces, another important consequence was that the generalized C-space photon dispersion relations allowed also for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary spacetimes, while breaking the *extended* Lorentz symmetry in C-spaces. This does *not* occur in DSR nor in other approaches, like the presence of quantum spacetime foam.

We learnt from Special Relativity that the concept of simultaneity is also relative. By the same token, we have shown in [4] that the concept of spacetime locality is *relative* due to the *mixing* of area-bivector coordinates with spacetime vector coordinates under generalized Lorentz transformations in C-space. In the most general case, there will be mixing of all polyvector valued coordinates. This was the motivation to build a unified theory of all extended objects, p-branes, for all values of p subject to the condition $p+1 = D$.

In [5] we explored the many novel physical consequences of Born's Reciprocal Relativity theory [7], [9], [10] in flat phase-space and generalized the theory to the curved phase-space scenario. We provided six specific novel physical results resulting from Born's Reciprocal Relativity and which are *not* present in Special Relativity. These were : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. We finalized by constructing a Born reciprocal general relativity theory in curved phase-spaces which required the introduction of a complex Hermitian metric, torsion and nonmetricity.

We should emphasize that *no* spacetime foam was introduced, nor Lorentz invariance was broken, in order to explain the time delay in the photon emission/arrival. In the conventional approaches of DSR (Double Special Relativity) where there is a Lorentz invariance breakdown [12], a longer wavelength photon (lower energy) experiences a smoother spacetime than a shorter wavelength photon (higher energy) because the higher energy photon experiences more of the graininess/foamy structure of spacetime at shorter scales. Consequently, the less energetic photons will move faster (less impeded) than the higher energetic ones and will arrive at earlier times.

However, in our case above [5] the time delay is entirely due to the very nature of Born's Reciprocal Relativity when one looks at pure acceleration (force) boosts transformations of the phase space coordinates in $flat$ phase-space. No *curved* momentum space is required as it happens in [12]. The time delay condition in Born's Reciprocal Relativity theory implied also that higher momentum (higher energy) photons will take longer to arrive than the lower momentum (lower energy) ones.

Superluminal particles were studied within the framework of the Extended Relativity theory in Clifford spaces (C-spaces) in [6]. In the simplest scenario, it was found that it is the contribution of the Clifford scalar component P of the poly-vector-valued momentum $\mathbf{P}$ which is responsible for the superluminal behavior in ordinary spacetime due to the fact that the effective mass $\sqrt{\mathcal{M}^2 - P^2}$ can be imaginary (tachyonic). However from the point of view of C-space there is no superluminal behaviour (tachyonic) because the true physical mass still obeys $\mathcal{M}^2 > 0$. As discussed in detailed by [1], [3] one can have tachyonic (superluminal) behavior in ordinary spacetime while having non-tachyonic behavior in C-space. Hence from the C-space point of view there is no violation of causality nor the Clifford-extended Lorentz symmetry. The analog of "photons" in C-space are *tensionless* strings and branes [1].

The addition law of areal velocities and a minimal length interpretation L was recently studied in [4]. The argument relied entirely on the physics behind the extended notion of Lorentz transformations in C-space, and *does not* invoke Quantum Gravity arguments nor quantum group deformations of Lorentz/Poincare algebras. The physics of the Extended Relativity theory in C-spaces requires the introduction of the speed of light and a minimal scale. In [2] we have shown how the construction of an Extended Relativity Theory in Clifford *Phase* Spaces requires the introduction of a *maximal* scale which can be identified with the Hubble scale and leads to Modifications of Gravity at the Planck/Hubble scales. Born's Reciprocal Relativity demands that a minimal length corresponds to a minimal momentum that can be set to be $p_{min} = \hbar / R_{Hubble}$. For full details we refer to [2].

Despite the fact that the length parameter L (which must be introduced in the C-space interval in eq-(1.2) in order to match units) has the physical interpretation of a *minimal* length, this does *not* mean that the spatial separation between two events in C-space cannot be *smaller* than L. The Planck scale minimal length argument is mainly associated with Quantum Mechanics and Black Hole Physics. The energy involved in the physical measurement process to localize a Planck mass particle, within Planck scale resolutions, becomes very large and such that a black hole forms enclosing the particle behind the black hole horizon. Since one does not have physical access to the black hole interior one cannot probe scales beyond the Planck scale. We shall set aside for the moment the current firewall controversy of black holes.

2 Stringy Uncertainty Relations from Clifford Spaces

The generalization of the Weyl-Heisenberg algebra to C-spaces and involving polyvector-valued coordinates and momenta (in natural units $\hbar = 1$) is [1]

$$[X_A,\ P_B] = \ i\ G_{AB} \tag{2.1}$$

and does not lead to minimal uncertainty conditions for ΔX_A. To obtain the minimal length stringy uncertainty relations in ordinary spacetimes requires more work. It involves taking polymomentum slices through C-space. This is the subject of this section.

The on-shell mass condition for a massive polyparticle moving in the 2^4-dimensional flat C-space, corresponding to a Clifford algebra in $D = 4$, can be written in terms of the polymomentum (polyvector-valued) components, in natural units $L = L_P = 1, \hbar = c = 1$, as

$$\pi^2 \ + \ p_\mu\ p^\mu \ + \ p_{\mu_1\mu_2}\ p^{\mu_1\mu_2} \ + \ p_{\mu_1\mu_2\mu_3}\ p^{\mu_1\mu_2\mu_3} \ + \ p_{\mu_1\mu_2....\mu_4} p^{\mu_1\mu_2...\mu_4} \ = \ \mathcal{M}^2 \tag{2.2}$$

Below we will argue why the π^2 term must appear with a negative sign due to the choice $G_{**} = -1 < 0$ of the scalar-scalar component of the C-space metric G_{AB}.

A particular *slice* through the flat C-space can be taken by imposing the set of algebraic conditions on the polymomenta coordinates

$$p_{\mu_1\mu_2}\ p^{\mu_1\mu_2} \ = \ \lambda_1\ (p_\mu p^\mu)^2 \ = \ \lambda_1\ p^4, \ \ p_{\mu_1\mu_2\mu_3}\ p^{\mu_1\mu_2\mu_3} \ = \ \lambda_2\ (p_\mu p^\mu)^3 \ = \ \lambda_2\ p^6$$

$$p_{\mu_1\mu_2\mu_3\mu_4}\ p^{\mu_1\mu_2\mu_3\mu_4} \ = \ \lambda_3\ (p_\mu p^\mu)^4 \ = \ \lambda_3\ p^8 \tag{2.3}$$

$$p^2 \ \equiv \ p_\mu\ p^\mu \ = \ |\vec{p}|^2 \ - \ (p_0)^2 \ = \ (p_x)^2 + (p_y)^2 + (p_z)^2 \ - \ E^2 \tag{2.4}$$

where the λ's are numerical parameters. π is the Clifford scalar part of the momentum polyvector and is invariant under C-space transformations. The slice conditions in eqs-(2.3) will *break* the generalized (extended) Lorentz symmetry in C-space because these conditions are *not* preserved under the most general C-space transformations as described in [4]. Nevertheless, the residual standard Lorentz symmetry (in ordinary spacetime) will still remain intact because the conditions/constraints in eqs-(2.3) are explicitly Lorentz invariant.

Inserting the conditions of eqs-(2.3) into eq-(2.2) yields

$$p^2 \; (\; \frac{\pi^2}{p^2} \; + \; 1 \; + \lambda_1 \; p^2 \; + \lambda_2 \; p^4 \; + \lambda_3 \; p^6 \;) \; = \; f(\pi^2, p^2) \; p^2 \; = \; \mathcal{M}^2 \qquad (2.5)$$

The expression for an infinitesimal interval in (curved) momentum space is

$$(d\sigma)^2 \; = \; g_{\mu\nu}(p^\rho) \; dp^\mu \; dp^\nu \; = \; g_{\mu\nu}(p^\rho) \; \frac{dp^\mu}{d\sigma} \; \frac{dp^\nu}{d\sigma} \; (d\sigma)^2 \qquad (2.6)$$

From (2.6) one infers that

$$g_{\mu\nu}(p^\rho) \; \frac{dp^\mu}{d\sigma} \; \frac{dp^\nu}{d\sigma} \; \equiv \; g_{\mu\nu}(p^\rho) \; \pi^\mu \; \pi^\nu \; = \; 1, \quad \pi^\mu \equiv \frac{dp^\mu}{d\sigma} \qquad (2.7)$$

The local coordinates in (curved) momentum space are $p^\rho = p^0, p^1, p^2, p^3$; whereas π^μ is a four-vector in (curved) momentum space.

Let us re-write the mass-shell condition (2.5) in terms of a curved momentum space metric as follows

$$g_{\mu\nu}(\pi^2, p^2) \; p^\mu \; p^\nu \; = \; g^{\mu\nu}(\pi^2, p^2) \; p_\mu \; p_\nu \; = \; \mathcal{M}^2 \qquad (2.8)$$

since the local coordinates p^μ are *not* to be confused with the components of the four-vector π^μ in (curved) momentum space, the mass-shell condition (2.5,2.8) $differs$ from the unit normalization condition (2.7) of π^μ. Nevertheless, one *can* still $define$ a momentum space metric as if it were *emergent* from having taken a slice in the C-space (described by eqs-(2.3)) as

$$g_{\mu\nu}(\pi^2, p^2) \; = \; f(\pi^2, p^2) \; \eta_{\mu\nu}, \quad g^{\mu\nu}(p^2) \; = \; \frac{1}{f(\pi^2, p^2)} \; \eta^{\mu\nu}, \quad \eta_{\mu\nu} = \eta^{\mu\nu} = diag(-1, 1, 1, 1). \qquad (2.9a)$$

from eq-(2.8) and the fact that $p^2 = \eta_{\mu\nu} \; p^\mu \; p^\nu = \eta^{\mu\nu} \; p_\mu \; p_\nu$, one can infer that

$$f(\pi^2, p^2) \; = \; \frac{1}{f(\pi^2, p^2)} \; \Rightarrow \; f(\pi^2, p^2) \; = \; (\; \frac{\pi^2}{p^2} \; + \; 1 \; + \lambda_1 \; p^2 \; + \lambda_2 \; p^4 \; + \lambda_3 \; p^6 \;) \; = \; 1 \qquad (2.9b)$$

where we disregard the $f(\pi^2, p^2) = -1$ solutions. The metric (2.9a) turns out to be flat since the conformal factor is constrained to unity. Eq-(2.9b) is the required C-space slice condition that π^2 must satisfy leading to a constraint among p^2 and π^2. In the same fashion eqs-(2.3) provide a constraint among p^2 and the other polymomentum components.

Since $g_{\mu\nu}(\pi^2, p^2) = f(\pi^2, p^2)\eta_{\mu\nu} \to \eta_{\mu\nu}$ when $f(\pi^2, p^2) = 1$ there is *no* modification to the Weyl-Heisenberg algebra $[x_\mu, \; p_\nu] = i\hbar \; \eta_{\mu\nu}$ and no modifications to the uncertainty relations A curved momentum space within the context of DSR has been studied by [12]. Finsler geometry is the proper arena to study metrics which depend on both coordinates and velocities/momenta.

If instead of the trivial metric $g_{\mu\nu}(\pi^2, p^2) = \eta_{\mu\nu}$ one has now a curved (modified) momentum space metric $\rho_{\mu\nu}(p^2)$ given by

$$\rho_{\mu\nu}(p^2) \;=\; g(p^2)\;\eta_{\mu\nu} \;=\; (\;1\;+\lambda_1\;p^2\;+\lambda_2\;p^4\;+\lambda_3\;p^6\;)\;\eta_{\mu\nu}\;\neq\;\eta_{\mu\nu}$$

a modified Weyl-Heisenberg algebra could be defined as follows

$$[x_\mu,\;p_\nu] \;=\; i\;\hbar\;\rho_{\mu\nu}(p^2)\;\equiv\; i\;\hbar_{eff}(p^2)\;\eta_{\mu\nu},\;\;\hbar_{eff}(p^2) \equiv \hbar\;g(p^2) \tag{2.10}$$

the above algebra can be recast in terms of an effective momentum-dependent Planck "constant" $\hbar_{eff}(p^2)$ defined by

$$\hbar_{eff}(p^2)\;\equiv\;\hbar\;g(p^2)\;=\;\hbar\;(\;1\;+\lambda_1\;p^2\;+\lambda_2\;p^4\;+\lambda_3\;p^6\;) \tag{2.11}$$

and which *emerged* from taking a slice in C-space displayed by eqs-(2.3).

In the most general case one must recur to a (curved) phase space and a *matrix*-valued Planck "constant" $\hbar_{\mu\nu}(x_\rho, p_\rho)$ which is a function of both coordinates and momenta. The most general Weyl-Heisenberg algebra is then given by

$$[x_\mu,\;p_\nu]\;=\;i\;\hbar_{\mu\nu}(x_\rho, p_\rho)\;=\;i\hbar\;\Theta_{\mu\nu}(x_\rho, p_\rho) \tag{2.12}$$

However, since one must obey the Jacobi identities among the commutators, one must have in the most general case that the coordinates and momenta must be noncommutative

$$[x_\mu,\;x_\nu]\;\neq\;0,\;\;[p_\mu,\;p_\nu]\;\neq\;0 \tag{2.13}$$

Next section will be devoted to the study and solutions to the Jacobi identities.

To simplify matters we shall choose $\Theta_{\mu\nu} = g(p^2)\eta_{\mu\nu}$ and $[p_\mu, p_\nu] = 0$ but $[x_\mu, x_\nu] \neq 0$ whose physical motivation lies in the fact that the tangent space to a curved-momentum space can be identified with spacetime. A flat spacetime (zero curvature) is compatible with commuting momentum $[p_\mu, p_\nu] = [i\hbar\nabla_{x^\mu}, i\hbar\nabla_{x^\nu}] = 0$. Whereas $[x_\mu, x_\nu] = [i\hbar\nabla_{p^\mu}, i\hbar\nabla_{p^\nu}] \neq 0$ is consistent with a non-zero curvature in momentum space.

A careful study of

$$[x_\mu,\;p_\nu]\;=\;i\;\hbar_{eff}(p^2)\;\eta_{\mu\nu}\;=\;i\;\;\hbar\;(\;1\;+\lambda_1\;p^2\;+\lambda_2\;p^4\;+\lambda_3\;p^6\;)\;\eta_{\mu\nu} \tag{2.14}$$

reveals that it does *not* lead to minimal length uncertainty relations. It is due to the crucial *minus* sign appearing in $p^2 = |\vec{p}|^2 - (p_0)^2$ which leads to a flip in the $\geq$ inequality symbol to one involving the $\leq$ inequality symbol. This will become clear below.

This is consistent with the fact that the ordinary Lorentz invariance was not broken by imposing the C-space slice conditions (2.3), (2.9b). It was the generalized extended Lorentz symmetry in C-space which was broken by imposing these conditions. Since Lorentz invariance was not broken, a Lorentz boost transformation leads to a length *contraction* which can be *zero* (no minimal length) when the Lorentz boost parameter is *infinite* (the new frame of reference moves at the speed of light).

However, matters will change drastically when one *breaks* the ordinary Lorentz invariance. This is attained by imposing the non-Lorentz invariant conditions on the poly-momenta in C-space

$$p_{ij}\ p^{ij} \ = \ \beta_1\ |\vec p|^4, \ \ p_{ijk}\ p^{ijk} \ = \ \beta_2\ |\vec p|^6 \qquad (2.15a)$$

$$p_{0i}\ p^{0i} \ = \ \alpha_1\ (p_0)^2\ |\vec p|^2, \ \ p_{0ij}\ p^{0ij} \ = \ \alpha_2\ (p_0)^2\ |\vec p|^4, \ \ p_{0ijk}\ p^{0ijk} \ = \ \alpha_3\ (p_0)^2\ |\vec p|^6 \quad (2.15b)$$

where the α's and β's are numerical parameters. The mass-shell condition (2.2) in C-space becomes after inserting the conditions (2.15) and taking into account the chosen signature $(-,+,+,+)$

$$|\vec p|^2\ (\ \frac{\pi^2}{|\vec p|^2}\ +1\ +\beta_1\ |\vec p|^2\ +\beta_2\ |\vec p|^4\ ) - (p_0)^2\ (\ 1\ +\alpha_1\ |\vec p|^2\ +\alpha_2\ |\vec p|^4\ +\ \alpha_3\ |\vec p|^6\ )\ =\ \mathcal{M}^2 \quad (2.16)$$

again, one may notice that the terms inside the parenthesis behave as if one had a metric in momentum space; namely one can rewrite the above equation (2.16) as follows

$$g_{ij}(\pi^2, |\vec p|^2)\ p^i\ p^j\ +\ g_{00}(|\vec p|^2)\ p^0\ p^0\ =\ g^{ij}(\pi^2, |\vec p|^2)\ p_i\ p_j\ +\ g^{00}(|\vec p|^2)\ p_0\ p_0\ =\ \mathcal{M}^2 \quad (2.17)$$

Following the same arguments as described in eqs-(2.8, 2.9a, 2.9b) one arrives at

$$g_{ij}(\pi^2, |\vec p|^2)\ =\ \delta_{ij}\ \Rightarrow\ \frac{\pi^2}{|\vec p|^2}\ +1\ +\beta_1\ |\vec p|^2\ +\beta_2\ |\vec p|^4\ =\ 1 \qquad (2.18)$$

which leads to a non-Lorentz invariant constraint among π^2 and $|\vec p|^2$. The former π^2 is a Lorentz scalar but not the latter. The other condition is

$$g_{00}(|\vec p|^2)\ =\ -1\ \Rightarrow\ -\ (\ 1\ +\alpha_1\ |\vec p|^2\ +\alpha_2\ |\vec p|^4\ +\ \alpha_3\ |\vec p|^6\ )\ =\ -1 \qquad (2.19)$$

from which one infers that the parameters $\alpha_1 = \alpha_2 = \alpha_3 = 0$ are zero because one should not impose constraints of the values of $|\vec p|^2$. Hence, having $\alpha_1 = \alpha_2 = \alpha_3 = 0$ in (2.15b) implies that the polymomentum slice in C-space will set the following values to zero : $p_{0i} = p_{0ij} = p_{0ijk} = 0$.

The *key* step in deriving the string uncertainty relations relies in the inequalities resulting directly from the condition in eq-(2.18)

$$1\ \geq\ 1\ +\beta_1\ |\vec p|^2\ +\beta_2\ |\vec p|^4, \ \ if\ \ \pi^2 \geq 0 \qquad (2.20a)$$

$$1\ \leq\ 1\ +\beta_1\ |\vec p|^2\ +\beta_2\ |\vec p|^4, \ \ if\ \ \pi^2 \leq 0 \qquad (2.20b)$$

The choice $\pi^2 > 0$ gives $\beta_1|\vec p|^2 + \beta_2|\vec p|^4 < 0$ forcing $\beta_2 < 0$ which leads to inconsistencies with the positive definite conditions of eq-(2.15a) for the chosen signatures, and forces constraints on the domain of values of $|\vec p|^2$. Therefore, one must choose $\pi^2 < 0$ that gives $\beta_1|\vec p|^2 + \beta_2|\vec p|^4 > 0$ for $\beta_1 > 0, \beta_2 > 0$ as required and does not introduce constraints on the domain of values of $|\vec p|^2$.

The immediate problem is that having $\pi^2 < 0$ leads to imaginary values for π. To solve this problem we must fix the choice for the sign of the scalar-scalar components G_{**} of the C-space metric G_{AB}. We shall fix $G_{**} = -1 < 0$ so that when one evaluates to the mass-shell condition (2.2) one will have the π^2 term with the required negative sign $G_{**}\pi^2 = -\pi^2 < 0$ so that eq-(2.18) should read

$$ - \frac{\pi^2}{|\vec{p}|^2} + 1 + \beta_1 |\vec{p}|^2 + \beta_2 |\vec{p}|^4 = 1 \qquad (2.18') $$

leading to proper inequality $1 \leq 1 + \beta_1 |\vec{p}|^2 + \beta_2 |\vec{p}|^4$ as required . The flat metric $g_{ij}(\pi^2, |\vec{p}|^2) = \delta_{ij}$ and $g_{00}(|\vec{p}|^2) = -1$ does not lead to modifications of the Weyl-Heisenberg algebra.

However, it is upon using the *key* inequality in eq-(2.20b), and treating the coordinates and momenta as self-adjoint quantum *operators*, which leads to the following uncertainty relations

$$ \Delta x_i \ \Delta p_j \ \geq \ \frac{\hbar}{2} \ | < (1 + \beta_1 |\vec{p}|^2 + \beta_2 |\vec{p}|^4) > | \ \delta_{ij} \ \geq \ \frac{\hbar}{2} \ \delta_{ij} \qquad (2.21a) $$

or

$$ \frac{\hbar}{2} \ | < (1 + \beta_1 |\vec{p}|^2 + \beta_2 |\vec{p}|^4) > | \ \delta_{ij} \ \geq \ \Delta x_i \ \Delta p_j \ \geq \ \frac{\hbar}{2} \ \delta_{ij} \ , \quad \beta_1 > 0, \ \beta_2 > 0 \qquad (2.21b) $$

where $< .... >$ denote the QM expectation values $< \Psi|......|\Psi >$. See [21] for rigorous mathematical details. One may notice that the inequalities in (2.21a) yields the stringy uncertainty relations, but the inequalities (2.21b) do *not*. Keeping the leading terms in powers of L_P in eqs-(2.21a), gives

$$ \Delta x \ \Delta p_x \ \geq \ \frac{\hbar}{2} \ | < \ (1 + \beta_1 |\vec{p}|^2) \ > | \ \geq \ \frac{\hbar}{2} \ | < \ (1 + \beta_1 \ p_x^2) \ > | \ \geq \ \frac{\hbar}{2} \ (1 + \beta_1 \ (\Delta p_x)^2) \qquad (2.22) $$

where we have used the identities $< p_x^2 >= (\Delta p_x)^2 + < p_x >^2$ in last inequality of (2.22), and taken $\beta_1 > 0$ which allows to remove the absolute sign since all quantities are now positive definite.

From (2.22) one arrives at the minimal length stringy uncertainty relations

$$ \Delta x \ \Delta p_x \ \geq \ \frac{\hbar}{2} \ (1 + \beta_1 \ (\Delta p_x)^2) \ \Rightarrow \ \Delta x \ \geq \ \frac{\hbar}{2\Delta p_x} \ + \ (\frac{\hbar\beta_1}{2}) \ \Delta p_x \qquad (2.23) $$

Minimizing the expression in (2.23) and inserting the Planck scale L_P which was set to unity one has for the minimum position uncertainty a quantity of the order of the Planck scale

$$ (\Delta x)_{min} \ = \ L_P \ \sqrt{\beta_1}, \quad \beta_1 > 0 \qquad (2.24) $$

following similar arguments for the other coordinates leads to

$$ \Delta y \ \Delta p_y \ \geq \ \frac{\hbar}{2} \ (1 + \beta_1 \ (\Delta p_y)^2) \ \Rightarrow \ \Delta y \ \geq \ \frac{\hbar}{2\Delta p_y} \ + \ (\frac{\hbar\beta_1}{2}) \ \Delta p_y \qquad (2.25) $$

$$ \Delta z \ \Delta p_z \ \geq \ \frac{\hbar}{2} \ (1 + \beta_1 \ (\Delta p_z)^2) \ \Rightarrow \ \Delta z \ \geq \ \frac{\hbar}{2\Delta p_z} \ + \ (\frac{\hbar\beta_1}{2}) \ \Delta p_z \qquad (2.26) $$

and a minimization procedure leads to the same values for the minimal length uncertainties for the other spatial coordinates $\Delta y = \Delta z = L_P \sqrt{\beta_1}$.

However, there is no modification to the energy-time commutators ($x_0 = ct$)

$$[x_0,\ p_0] \ = \ i\ \hbar\ g_{00}(|\vec{p}|^2) \ = \ -\ i\ \hbar \tag{2.27}$$

as a result of eq-(2.19). Therefore, Δx_0 does not admit a minimum uncertainty (besides the trivial case $\Delta x_0 = 0$) since it obeys $\Delta x_0\ \Delta p_o \geq \frac{\hbar}{2}$.

An important remark is in order. Note that one cannot minimize, simultaneously, the three following expressions

$$\Delta x_i \ \geq \ \frac{\hbar}{2\Delta p_i}\ [\ 1\ +\ \beta_1\ (\ (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2\ )\ ], \quad i = 1, 2, 3. \tag{2.28}$$

In eqs-(2.23, 2.25, 2.26) we have minimized the Δx with $\Delta p_x \neq 0$ and $\Delta p_y = \Delta p_z = 0$. Minimized Δy with $\Delta p_y \neq 0$ and $\Delta p_x = \Delta p_z = 0$. And minimized Δz with $\Delta p_z \neq 0$ and $\Delta p_x = \Delta p_y = 0$. However, there is *no* common set S of solutions for $S = \{\Delta p_x, \Delta p_y, \Delta p_z\}$ which minimizes $\Delta x, \Delta y, \Delta z$ in the above *three* eqs-(2.28) simultaneously.

The *higher* order corrections in eq-(2.21) stem from the higher grade polymomentum variables in C-space and correspond, physically, to the membrane contributions to the modified uncertainty relations. Hence, the stringy and membrane corrections to the uncertainty relations in $D = 4$ are of the form (similar equations follow for the other spatial coordinates)

$$\Delta x\ \Delta p_x \ \geq \ \frac{\hbar}{2}\ [\ 1\ + \beta_1\ (\Delta p_x)^2 \ +\ \beta_2\ (\Delta p_x)^4\ ] \tag{2.29}$$

leading to

$$\Delta x \ \geq \ \frac{\hbar}{2}\ [\ \frac{1}{\Delta p_x}\ + \beta_1\ (\Delta p_x) \ +\ \beta_2\ (\Delta p_x)^3\ ] \tag{2.30}$$

the extremization problem of (2.20) is more complicated but there is a local minimum when $\beta_1 > 0, \beta_2 > 0$. The value of Δp_x which yields the local minimum for Δx is

$$(\Delta p_x)_o \ = \ \left(\frac{-\ \beta_1\ +\ \sqrt{(\beta_1)^2 + 12\beta_2}}{6\beta_2} \right)^{\frac{1}{2}}, \quad \beta_1 > 0,\ \beta_2 > 0 \tag{2.30}$$

In higher dimensions than $D = 4$ one will capture the p-brane contributions beyond the membrane case due to the contributions of the higher grade polymomenta components. The dimensions (units) of the parameters in eqs-(2.29, 2.30) are $[\beta_1] = (L/\hbar)^2,\ [\beta_2] = (L/\hbar)^4$.

3 Jacobi Identities and Noncommutative Spacetime

To finalize we study the Jacobi identities that are linked to noncommuting spacetime coordinates. By invoking the fact that x_i, p_j are local coordinates, meaning that one raises/lowers indices with δ^{ij}, δ_{ij}, respectively, it is convenient to rewrite the modified Weyl-Heisenberg algebra as

$$[x^i,\ p_j]\ =\ [\delta^{ik}\ x_k,\ p_j]\ =\ i\ \hbar_{eff}(|\vec{p}|^2)\ \delta^{ik}\ \delta_{kj}\ =\ i\ \hbar_{eff}(|\vec{p}|^2)\ \delta^i_j\ =\ i\ \hbar\ g(|\vec{p}|^2)\ \delta^i_j \quad (3.1a)$$

$$g(|\vec{p}|^2)\ \equiv\ (\ 1\ +\beta_1\ |\vec{p}|^2\ +\beta_2\ |\vec{p}|^4\ ) \quad (3.1b)$$

The Jacobi identities are

$$[\ x^i,\ [x^j,\ p_k]\ ]\ +\ [\ x^j,\ [p_k,\ x^i]\ ]\ +\ [\ p_k,\ [x^i,\ x^j]\ ]\ =\ 0 \quad (3.2)$$

$$[\ x^i,\ [x^j,\ x^k]\ ]\ +\ [\ x^j,\ [x^k,\ x^i]\ ]\ +\ [\ x^k,\ [x^i,\ x^j]\ ]\ =\ 0, \quad (3.3)$$

etc, ... Let us try the ansatz

$$[x^i,\ x^j]\ =\ i\ \hbar\ f^{ij}{}_l(\vec{p})\ x^l,\quad [p_j,\ p_k]\ =\ 0 \quad (3.4)$$

due to the noncommutativity of x^i, p^j one could have written instead of (3.4) the following more symmetric form for the commutators

$$[x^i,\ x^j]\ =\ \frac{i\hbar}{2}\ \{f^{ij}{}_l(\vec{p}),\ x^l\}\ =\ \frac{i\hbar}{2}\ f^{ij}{}_l(\vec{p})\ x^l\ +\ \frac{i\hbar}{2}\ x^l\ f^{ij}{}_l(\vec{p}) \quad (3.5)$$

For simplicity, we will just use the commutators displayed in eq-(3.4) instead of those in eq-(3.5). It will not affect the final results. After some straightforward algebra one learns from the Jacobi identities (3.2) that the structure functions $f^{ij}{}_l(\vec{p})$ are given in terms of the function $g(|\vec{p}|^2)$ given by (3.1b) as follows

$$\delta^j_k\ \frac{\partial g(|\vec{p}|^2)}{\partial p_i}\ -\ \delta^i_k\ \frac{\partial g(|\vec{p}|^2)}{\partial p_j}\ =\ f^{ij}{}_k(\vec{p}) \quad (3.6)$$

it is explicitly antisymmetric in ij as expected. Using the second set of Jacobi identities (3.3) for the noncommutative spacetime coordinates, the relations $[x^i, F(\vec{p})] = i\hbar_{eff}(|\vec{p}|^2)\ (\partial F(\vec{p})/\partial p_i)$, the Liebnitz law $[x^i, AB] = A[x^i, B] + [x^i, A]B$, and the solutions obtained for $f^{ij}{}_k(\vec{p})$ given in (3.6), one can verify, after some algebra, that indeed one has

$$(\ f^{jk}{}_l\ f^{il}{}_m\ +\ f^{ki}{}_l\ f^{jl}{}_m\ +\ f^{ij}{}_l\ f^{kl}{}_m\ )\ x^m\ =\ 0 \quad (3.7)$$

$$(\ \frac{\partial f^{jk}{}_l}{\partial p_i}\ +\ \frac{\partial f^{ki}{}_l}{\partial p_j}\ +\ \frac{\partial f^{ij}{}_l}{\partial p_k}\ )\ x^l\ =\ 0 \quad (3.8)$$

and the Jacobi identities (3.2,3.3) are satisfied. It is important to emphasize that the terms inside the parenthesis in eqs-(3.7,3.8) are *not* zero. What *is* zero is the net summation after the full contraction with the x^m, x^l terms is performed.

Therefore, to satisfy the Jacobi identities one must have a Noncommutative spacetime. Kempf and Mangano [21] used the commutator $[x_i, p_j] = i\hbar\Theta_{ij}(\vec{p})$, where Θ_{ij} is a more general rotationally invariant function of the momenta coordinates, and for commutator $[x_i, x_j]$ they have the more symmetric expression described by (3.5). After studying the Jacobi identities they arrived at

$$[x_i, x_j] \;=\; i\hbar\ \{\ x_a,\ \Theta^{-1}_{ar}\ \Theta_{s[i}\ \Theta_{j]r,s}\ \},\quad \Theta_{jr,s}\ \equiv\ \frac{\partial\Theta_{jr}}{\partial p^s} \tag{3.9}$$

where $\{,\}$ denones the anti-commutator. See [21] for further details.

The theory of Scale Relativity proposed by Nottale [11] is based on a minimal observational length-scale, the Planck scale, as there is in Special Relativity a maximum speed, the speed of light, and deserves to be looked within the Clifford algebraic perspective. We conclude by pointing out that in the quantization program a key role must be played by quantum Clifford-Hopf algebras since the latter q-Clifford algebras naturally contain the κ-deformed Poincare algebras [16], [17], which are essential ingredients in the formulation of DSR within the context of Noncommutative spaces. The Minkowski spacetime quantum Clifford algebra structure associated with the conformal group and the Clifford-Hopf alternative κ-deformed quantum Poincare algebra was investigated [15]. The resulting algebra is equivalent to the deformed anti-de Sitter algebra $U_q(so(3,2))$, when the associated Clifford-Hopf algebra is taken into account, together with the associated quantum Clifford algebra and a (not braided) deformation of the periodicity Atiyah-Bott-Shapiro theorem [19].

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