The Arm Prime Factors Decomposition

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Abstract
We introduce the Arm prime factors decomposition which is the equivalent of the Taylor formula for decomposition of integers on the basis of prime numbers. We make the link between this decomposition and the p-adic norm known in the p-adic numbers theory. To see how it works, we give examples of these two formulas.

Introduction

The Arm theory [1] gives the decomposition of functions on each function basis. Also, because the fundamental arithmetic theorem tells us each integers can be decompose in prime factors, I wondered if it was possible to build the decomposition of integers on the basis of prime integers.

As a matter of fact, this is possible and the answer of this question is the Arm prime factors decomposition. To build an equivalent of the Taylor formula for the prime numbers basis, we need to find a good scalar product on this basis.

In first place, I remarked that if a prime number $p \in \mathbb{P}$ divise an integer $n \in \mathbb{N}^{+*}$ then the difference between it and its integer part would be zero whereas that if p does not divise n then the same difference would be positive. In mathematical words it means :

$$\begin{aligned} \text{if} \quad p|n \quad \text{then} \quad \frac{n}{p} - \left[\frac{n}{p}\right] &= 0 \\ \text{otherwise} \quad \frac{n}{p} - \left[\frac{n}{p}\right] &> 0 \end{aligned} \tag{0.1}$$

However, we need to build a Kronecker which is one when $\frac{n}{p}$ is an integer and zero otherwise. This is why we take the exponential minus (0.1) multiply by x :

$$"\delta_{p|n}" = \lim_{x\to\infty} \exp\left(- \left(\frac{n}{p} - \left[\frac{n}{p}\right] \right) x \right) \tag{0.2}$$

which is one when p divise n and zero otherwhise.

For all that, it still remains the problem of the multiplicity m_i of each prime number p_i because (0.2) tells us only if a prime number p is in the prime factors decomposition of an integer n. Thus we have to sum all the power of p if we want to know the multiplicity of each factor :

$$\sum_{m=1}^{\infty} \delta_{p^m|n} = m_i \tag{0.3}$$

With all these ingredients, we build the final Arm prime factors decomposition formula (1.5) which is nothing else that the multiplication of (0.3) by the logarithm of all prime numbers .

In this case the formula (0.3) gives a p-adic valuation in the p-adic number theory. Thus we can give an explicit expression for the well-known p-adic norm of rational numbers used in the p-adic numbers theory :

$$\left|\left| \frac{a}{b} \right|\right|_p = p^{\sum_{m=1}^{\infty} \left(\delta_{p^m|b} - \delta_{p^m|a} \right)} \tag{0.4}$$

for each prime numbers $p \in \mathbb{P}$.

In the first section, we introduce the Arm prime factors decomposition formula which gives the decomposition of each integer in the prime numbers basis. After describing the scalar product, we give in the second section a full example of prime factors decomposition with the number 277945762500 which is nothing else that $1^2.2^2.3^3.5^5.7^7$. Next in the third section, we give the formula of the p-adic norm and, in the fourth section, we calculate all p-adic norm of the rational $\frac{63}{550}$ which is $2^{-1}.3^2.5^{-2}.7.11^{-1}$.

1 The Arm Prime Factors Decomposition Formula

We give the projection on the prime integers basis of each positive integer n

Theorem 1. $\forall n \in \mathbb{N}^{+*}$ *the Arm prime factors decomposition is given by*

$$\ln(n) = \sum_{p\in\mathbb{P}} \left(\lim_{x\to\infty} \sum_{m=1}^{\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right)x\right) \right) \ln(p) \tag{1.5}$$

where $[\ ]$ *is the integer part and* $\mathbb{P}$ *is the set of all prime numbers.*

Proof :

The main idea here is that if $p^m|n$ (i.e. p^m divise n) then $\frac{n}{p^m} \in \mathbb{N}^{+*}$ and $\frac{n}{p^m} \in \mathbb{Q}^+ \setminus \mathbb{N}^+$. In other words, we have that if $p^m|n$ then $\frac{n}{p^m}$ is an integer so it is equal to its integer parts.
Because $\frac{n}{p^m} - \left[\frac{n}{p^m}\right] \geq 0$, we can construct the Kronecker $\forall m \in \mathbb{N}$ "$\delta_{p^m|n}$" :

$$\begin{aligned} \lim_{x\to\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right)x\right) &= 1 \quad \text{if} \quad p^m|n \\ \lim_{x\to\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right)x\right) &= 0 \quad \text{otherwise} \end{aligned} \tag{1.6}$$

$$\tag{1.7}$$

If we consider the usual prime factors decomposition showed in the fundamental theorem of arithmetic :

$$n = \prod_{i=1}^{\infty} (p_i)^{m_i} \tag{1.8}$$

where m_i is the multiplicity of each prime factors $p_i \in \mathbb{P}$.
In the Arm prime factors decomposition formula, the multiplicity m_i of each prime factor p_i is obtained by summing the Kronecker :

$$\sum_{m=1}^{\infty} \lim_{x\to\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right)x\right) = m_i \tag{1.9}$$

With (1.9), we can find the Arm prime factors decomposition formula (1.5) :

$$\begin{aligned} \ln(n) &= \sum_i m_i \ \ln(p_i) \\ &= \sum_i \left(\sum_{m=1}^{\infty} \lim_{x\to\infty} \exp\left(-\left(\frac{n}{p_i^m} - \left[\frac{n}{p_i^m}\right]\right)x\right) \right) \ln(p_i) \\ &= \sum_{p\in\mathbb{P}} \left(\sum_{m=1}^{\infty} \lim_{x\to\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right)x\right) \right) \ln(p) \\ \ln(n) &= \sum_{p\in\mathbb{P}} \left(\lim_{x\to\infty} \sum_{m=1}^{\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right)x\right) \right) \ln(p) \end{aligned} \tag{1.10}$$

which is the final result.

♦

Remark 1. *We can deduce of the Arm prime factors decomposition formula (1.5) that the scalar product of the logarithm of each integer* $n \in \mathbb{N}^{+*}$ *on the basis* $\{\ln p\}_{p\in\mathbb{P}}$ *is given by*

$$< \ln n, \ln p > \;=\; \lim_{x\to\infty} \sum_{m=1}^{\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right) x \right) \tag{1.11}$$

which the multiplicity of each prime factor.

Corollary 1. *Each integer* $n \in N^{+*}$ *can be decomposed in :*

$$n \;=\; \prod_{p\in\mathbb{P}} p^{\lim_{x\to\infty} \sum_{m=1}^{\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right)x\right)} \tag{1.12}$$

__**Proof :**__

We just take the exponential of (1.5)

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2 Example Of Arm Prime Factors decomposition

Here we apply the Arm prime factors decompostion formula to find the prime factors of the integer :

$$n \;=\; 277945762500 \tag{2.13}$$

The formula (1.5) gives :

$$\begin{aligned}
\ln(n) \;=\; & \ln(2) \lim_{x\to\infty} \left(e^{-(\frac{n}{2}-[\frac{n}{2}])x} + e^{-(\frac{n}{2^2}-[\frac{n}{2^2}])x} + e^{-(\frac{n}{2^3}-[\frac{n}{2^3}])x} + ... \right) \\
& + \ln(3) \lim_{x\to\infty} \left(e^{-(\frac{n}{3}-[\frac{n}{3}])x} + e^{-(\frac{n}{3^2}-[\frac{n}{3^2}])x} + e^{-(\frac{n}{3^3}-[\frac{n}{3^3}])x} + e^{-(\frac{n}{3^4}-[\frac{n}{3^4}])x} + ... \right) \\
& + \ln(5) \lim_{x\to\infty} \left(e^{-(\frac{n}{5}-[\frac{n}{5}])x} + e^{-(\frac{n}{5^2}-[\frac{n}{5^2}])x} + e^{-(\frac{n}{5^3}-[\frac{n}{5^3}])x} \right. \\
& \qquad \left. + e^{-(\frac{n}{5^4}-[\frac{n}{5^4}])x} + e^{-(\frac{n}{5^5}-[\frac{n}{5^5}])x} + e^{-(\frac{n}{5^6}-[\frac{n}{5^6}])x} + ... \right) \\
& + \ln(7) \lim_{x\to\infty} \left(e^{-(\frac{n}{7}-[\frac{n}{7}])x} + e^{-(\frac{n}{7^2}-[\frac{n}{7^2}])x} + e^{-(\frac{n}{7^3}-[\frac{n}{7^3}])x} + e^{-(\frac{n}{7^4}-[\frac{n}{7^4}])x} \right. \\
& \qquad \left. + e^{-(\frac{n}{7^5}-[\frac{n}{7^5}])x} + e^{-(\frac{n}{7^6}-[\frac{n}{7^6}])x} + e^{-(\frac{n}{7^7}-[\frac{n}{7^7}])x} + e^{-(\frac{n}{7^8}-[\frac{n}{7^8}])x} + ... \right) && (2.14) \\
& \vdots && (2.15)
\end{aligned}$$

which is with the value of n :

$$\begin{aligned}
\ln(n) &= \ln(2)\lim_{x\to\infty}\Big(e^{-(138972881250-[138972881250])x}+e^{-(69486440625-[69486440625])x}\\
&\qquad +e^{-(34743220312.5-[34743220312.5])x}+...\Big)\\
&\quad +\ln(3)\lim_{x\to\infty}\Big(e^{-(92648587500-[92648587500])x}+e^{-(30882862500-[30882862500])x}\\
&\qquad +e^{-(10294287500-[10294287500])x}+e^{-(3431429166.66-[3431429166.66])x}+...\Big)\\
&\quad +\ln(5)\lim_{x\to\infty}\Big(e^{-(55589152500-[55589152500])x}+e^{-(11117830500-[11117830500])x}+e^{-(2223566100-[2223566100])x}\\
&\qquad +e^{-(444713220-[444713220])x}+e^{-(88942644-[88942644])x}+e^{-(17788528.8-[1.77885288])x}+...\Big)\\
&\quad +\ln(7)\lim_{x\to\infty}\Big(e^{-(39706537500-[39706537500])x}+e^{-(5672362500-[5672362500])x}+e^{-(810337500-[810337500])x}\\
&\qquad +e^{-(115762500-[115762500])x}+e^{-(16537500-[16537500])x}\\
&\qquad +e^{-(2362500-[2362500])x}+e^{-(337500-[337500])x}+e^{-(48214.3-[48214.3])x}+...\Big)\\
&\quad \vdots
\end{aligned} \tag{2.16}$$

When we evaluate, it becomes :

$$\begin{aligned}
\ln(n) &= \ln(2)\lim_{x\to\infty}\Big(1+1+e^{-\frac{x}{2}}+...\Big)\\
&\quad +\ln(3)\lim_{x\to\infty}\Big(1+1+1+e^{-(0.666..)x}+...\Big)\\
&\quad +\ln(5)\lim_{x\to\infty}\Big(1+1+1+1+1+e^{-(0.8)x}+...\Big)\\
&\quad +\ln(7)\lim_{x\to\infty}\Big(1+1+1+1+1+1+1+e^{-(0.3)x}+...\Big)\\
&\quad \vdots
\end{aligned} \tag{2.17}$$

So we have the decomposition of $\ln(n)$ in prime factors :

$$\ln(n) = 2\ln(2)+3\ln(3)+5\ln(5)+7\ln(7) \tag{2.18}$$

Hence

$$n = 1^1.2^2.3^3.5^5.7^7 = \Upsilon(7) \tag{2.19}$$

If we define the integer function

$$\Upsilon(i) = \prod_{k\in\mathbb{P};k\le i} k^k \tag{2.20}$$

3 Link with the p-adic numbers

The formula (1.9) of an integer n gives the multiplicity of an prime factor. This multiplicity is called the p-adic valuation in the p-adic numbers theory. We give here its explicit expression

Proposition 1. *The p-adic valuation of a prime factor p_i of an integer n is given by*

$$v_p(n) \;=\; \sum_{m=1}^{\infty} \lim_{x\to\infty} \exp\left(-\left(\frac{n}{p^m} - \left[\frac{n}{p^m}\right]\right) x \right) \tag{3.21}$$

Proof :
See (1.9).

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Now, with the p-adic valuation, we can define the corresponding p-adic norm.

Proposition 2. *The p-adic norm of a rational number $\frac{a}{b} \in \mathbb{Q}$, where a and b are coprime, is given by*

$$\left|\left| \frac{a}{b} \right|\right|_p \;=\; p^{\lim_{x\to\infty} \sum_{m=1}^{\infty} \left(\exp\left(-\left(\frac{b}{p^m} - \left[\frac{b}{p^m}\right]\right)x\right) - \exp\left(-\left(\frac{a}{p^m} - \left[\frac{a}{p^m}\right]\right)x\right)\right)} \tag{3.22}$$

or its logarithm is :

$$\ln\left(\left|\left| \frac{a}{b} \right|\right|_p \right) \;=\; \left(\lim_{x\to\infty} \sum_{m=1}^{\infty} \left(\exp\left(-\left(\frac{b}{p^m} - \left[\frac{b}{p^m}\right]\right)x\right) - \exp\left(-\left(\frac{a}{p^m} - \left[\frac{a}{p^m}\right]\right)x\right)\right)\right) \ln\left(p\right) \tag{3.23}$$

Proof :
The p-adic norm of an integer a :

$$a \;=\; \prod_i \left(p_i\right)^{m_i} \tag{3.24}$$

as in (1.8), where the prime factor $p_i \in \mathbb{P}$ and m_i is the multiplicity of each p_i, is defined as

$$||\, a\, ||_{p_i} \;=\; p_i^{-m_i} \tag{3.25}$$

However we know from (1.9) that the multiplicity of each prime factor is given by p_i

$$\lim_{x\to\infty} \sum_{m=1}^{\infty} \exp\left(-\left(\frac{a}{p^m} - \left[\frac{a}{p^m}\right]\right)x \right) \;=\; m_i \tag{3.26}$$

Hence the p-adic norm of n (3.25) is given by

$$||\, a\, ||_{p_i} \;=\; p_i^{-\lim_{x\to\infty} \sum_{m=1}^{\infty} \exp\left(-\left(\frac{a}{p^m} - \left[\frac{a}{p^m}\right]\right)x\right)} \tag{3.27}$$

And so the p-adic norm of a rational $\frac{a}{b} \in \mathbb{Q}$ is given by

$$\left|\left| \frac{a}{b} \right|\right|_{p_i} \;=\; p_i^{\lim_{x\to\infty} \left(\sum_{m=1}^{\infty} \exp\left(-\left(\frac{b}{p^m} - \left[\frac{b}{p^m}\right]\right)x\right) - \exp\left(-\left(\frac{a}{p^m} - \left[\frac{a}{p^m}\right]\right)x\right) \right)} \tag{3.28}$$

which gives the formula (3.22).

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4 Example of p-adic norm

As example, we consider the rational :

$$\frac{a}{b} = \frac{63}{550} \tag{4.29}$$

With the formula (3.22), we can first calculate the 2-adic norm

$$\begin{aligned}
\ln\Big(\Big|\Big| \frac{63}{550} \Big|\Big|_2 \Big) &= \Big(\lim_{x\to\infty} \sum_{m=1}^{\infty} \Big(\exp\Big(-\Big(\frac{550}{2^m} - \Big[\frac{550}{2^m}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{2^m} - \Big[\frac{63}{2^m}\Big]\Big) x\Big)\Big) \ \ln(2) \\
&= \Big(\lim_{x\to\infty} \Big(\exp\Big(-\Big(\frac{550}{2} - \Big[\frac{550}{2}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{2} - \Big[\frac{63}{2}\Big]\Big) x\Big)\Big) \ \ln(2) \\
\ln\Big(\Big|\Big| \frac{63}{550} \Big|\Big|_2 \Big) &= \ln(2) \qquad (4.30)
\end{aligned}$$

or in the exponential form :

$$\Big|\Big| \frac{63}{550} \Big|\Big|_2 = 2 \tag{4.31}$$

Now we calculate the 3-adic norm

$$\begin{aligned}
\ln\Big(\Big|\Big| \frac{63}{550} \Big|\Big|_3 \Big) &= \Big(\lim_{x\to\infty} \sum_{m=1}^{\infty} \Big(\exp\Big(-\Big(\frac{550}{3^m} - \Big[\frac{550}{3^m}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{3^m} - \Big[\frac{63}{3^m}\Big]\Big) x\Big)\Big) \ \ln(3) \\
&= \Big(\lim_{x\to\infty} \Big(\exp\Big(-\Big(\frac{550}{3} - \Big[\frac{550}{3}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{3} - \Big[\frac{63}{3}\Big]\Big) x\Big)\Big) \ \ln(3) \\
&\quad + \Big(\lim_{x\to\infty} \Big(\exp\Big(-\Big(\frac{550}{9} - \Big[\frac{550}{9}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{9} - \Big[\frac{63}{9}\Big]\Big) x\Big)\Big) \ \ln(3) \\
\ln\Big(\Big|\Big| \frac{63}{550} \Big|\Big|_3 \Big) &= -2 \ \ln(3) \qquad (4.32)
\end{aligned}$$

or in the exponential form :

$$\Big|\Big| \frac{63}{550} \Big|\Big|_3 = \frac{1}{9} \tag{4.33}$$

Now we calculate the 5-adic norm

$$\begin{aligned}
\ln\Big(\Big|\Big| \frac{63}{550} \Big|\Big|_5 \Big) &= \Big(\lim_{x\to\infty} \sum_{m=1}^{\infty} \Big(\exp\Big(-\Big(\frac{550}{5^m} - \Big[\frac{550}{5^m}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{5^m} - \Big[\frac{63}{5^m}\Big]\Big) x\Big)\Big) \ \ln(5) \\
&= \Big(\lim_{x\to\infty} \Big(\exp\Big(-\Big(\frac{550}{5} - \Big[\frac{550}{5}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{5} - \Big[\frac{63}{5}\Big]\Big) x\Big)\Big) \ \ln(5) \\
&\quad + \Big(\lim_{x\to\infty} \Big(\exp\Big(-\Big(\frac{550}{25} - \Big[\frac{550}{25}\Big]\Big) x\Big) - \exp\Big(-\Big(\frac{63}{25} - \Big[\frac{63}{25}\Big]\Big) x\Big)\Big) \ \ln(5) \\
\ln\Big(\Big|\Big| \frac{63}{550} \Big|\Big|_5 \Big) &= 2 \ \ln(5) \qquad (4.34)
\end{aligned}$$

or in the exponential form :

$$\Big|\Big| \frac{63}{550} \Big|\Big|_5 = 25 \tag{4.35}$$

Now we calculate the 7-adic norm

$$\ln\Big(\ \Big|\Big|\ \frac{63}{550}\ \Big|\Big|_7\ \Big) = \Big(\lim_{x\to\infty}\sum_{m=1}^{\infty}\ \Big(\exp\Big(-\Big(\frac{550}{7^m}-\Big[\frac{550}{7^m}\Big]\Big)x\Big)-\exp\Big(-\Big(\frac{63}{7^m}-\Big[\frac{63}{7^m}\Big]\Big)x\Big)\Big)\ \ln(7)$$

$$= \Big(\lim_{x\to\infty}\ \Big(\exp\Big(-\Big(\frac{550}{7}-\Big[\frac{550}{7}\Big]\Big)x\Big)-\exp\Big(-\Big(\frac{63}{7}-\Big[\frac{63}{7}\Big]\Big)x\Big)\Big)\ \ln(7)$$

$$\ln\Big(\ \Big|\Big|\ \frac{63}{550}\ \Big|\Big|_7\ \Big) = -\ \ln(7) \tag{4.36}$$

or in the exponential form :

$$\Big|\Big|\ \frac{63}{550}\ \Big|\Big|_7 = \frac{1}{7} \tag{4.37}$$

Now we calculate the 11-adic norm

$$\ln\Big(\ \Big|\Big|\ \frac{63}{550}\ \Big|\Big|_{11}\ \Big) = \Big(\lim_{x\to\infty}\sum_{m=1}^{\infty}\ \Big(\exp\Big(-\Big(\frac{550}{11^m}-\Big[\frac{550}{11^m}\Big]\Big)x\Big)-\exp\Big(-\Big(\frac{63}{11^m}-\Big[\frac{63}{11^m}\Big]\Big)x\Big)\Big)\ \ln(11)$$

$$= \Big(\lim_{x\to\infty}\ \Big(\exp\Big(-\Big(\frac{550}{11}-\Big[\frac{550}{11}\Big]\Big)x\Big)-\exp\Big(-\Big(\frac{63}{11}-\Big[\frac{63}{11}\Big]\Big)x\Big)\Big)\ \ln(11)$$

$$\ln\Big(\ \Big|\Big|\ \frac{63}{550}\ \Big|\Big|_{11}\ \Big) = \ln(11) \tag{4.38}$$

or in the exponential form :

$$\Big|\Big|\ \frac{63}{550}\ \Big|\Big|_{11} = 11 \tag{4.39}$$

And we have that

$$\Big|\Big|\ \frac{63}{550}\ \Big|\Big|_p = 1 \tag{4.40}$$

$\forall p \neq \{2,3,5,7,11\}$. So we have the prime factors decomposition of $\frac{a}{b}$

$$\frac{63}{550} = 2^{-1}.3^2.5^{-2}7.11^{-1} \tag{4.41}$$

Discussion

Maybe the Arm prime factors formula (1.5) is too simple but it gives a practical way to calculate the decomposition of every logarithms of integers on the basis of the logarithm of prime numbers.

However, the Arm prime factors decomposition formula (1.5) is not efficient when it is programmed on computers because it does the same work as the traditional algorithm (it checks if the division is an integer or not). In addition, the traditional algorithm which do the prime factorization is faster than this one. The only utility of my formula is that it gives a practical formula in the theory.

Furthermore in the formula (1.5) there is summation over positive integers and we have to stop it until a big value if we want the algorithm to be finish. Besides there is an other summation over prime numbers and the algorithm needs to calculate all the prime numbers so it takes a lot of time for calculating. After all, I 've decided to write this article even if the algorithm is not efficient because the formulas are right and give the results.

Références

[1] Arm B. N., The Arm Theory