The Arm Lie Group Theory

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Abstract
We developp the Arm-Lie group theory which is a theory based onthe exponential of a changing of matrix variable $u(X)$. We define a corresponding u-adjoint action, the corresponding commutation relations in the Arm-Lie algebra and the u-Jacobi identity. Throught the exponentiation, Arm-Lie algebras become Arm-Lie groups. We give the example of $\sqrt[u]{\mathfrak{so}(2)}$ and $\sqrt[u]{\mathfrak{su}(2)}$.

Introduction

The Arm theory [1] gives the generaized Taylor formula in any basis. It give rise to exponentials of any changing of variable $u(x)$. While the Lie group theory has been built on the classical exponential, I wondered myself why not building a new Lie group theory based on exponential of changing of variable, i.e. on exponential :

$$e^{u(X)} \tag{0.1}$$

where $u(X)$ is any changing of variable.

Besides, we can build an new adjoint action with this exponential in a new basis. This is the 'u-adjoint action' given in proposition 1 :

$$\mathrm{ad}_u A.X \;=\; [A, u(X)] \tag{0.2}$$

The traditional Lie algebra satisfyied some commutation relations between their generators so I searched the corresponding commutation relations for this new structure. In fact the genrators of the Arm-Lie algebras $\{X_1, ..., X_n, u(X_1), ..., u(X_n)\}$, if n is the dimension, satisfy the following conditions

$$\begin{aligned} [X_i, u(X_j)] &= {\lambda_{ij}}^k u(X_k) \\ [X_i, X_j] &= {\nu_{ij}}^k u(X_k) \end{aligned} \tag{0.3}$$

with ${\lambda_{ij}}^k$ and ${\nu_{ij}}^k$ the corresponding u-structure constant and structure constant respectively. In addition, there is a corresponding u-Jacobi identity in Arm-Lie algebras which is given by :

$$[u(X), [Y, Z]] + [u(Y), [Z, X]] + [u(Z), [X, Y]] = 0 \tag{0.4}$$

In the same way, Lie algebras become Lie groups throught exponentiation, we can take the exponential of each linear combinations which gives a group that we call a Arm-Lie group. This is why we check in the third section that the commutator of commutators of elements in the Arm-Lie algebras are still in the Arm-Lie algebra.

However, the big default of my construction is that I only find one example of the Arm-Lie algebra, but what a beautiful example : $u^{-1}(\mathfrak{so}(2))$ and $u^{-1}(\mathfrak{su}(2))$ with $\forall p \in \mathbb{C}, u(x) = X^p$. I think it is because the genrators of $\mathfrak{su}(2)$ and $\mathfrak{so}(2)$ are the exponential of something (i.e. their logarithm exist) but I am not sure yet. Nevertheless I hope there is other examples of Arm-Lie algebras. Moreover, the exponential of those two Arm-Lie algebras give two new groups which are Arm-Lie groups that we call $rSO(2)$ and $rSU(2)$ given by

$$rSO(2, \mathbb{C}) = \left\{ M \in M_l(\mathbb{C}) \;|\; {}^t MM \;= r\mathrm{Id}_2 \;;\; r \in \mathbb{C} \right\} \tag{0.5}$$

and

$$rSU(2, \mathbb{C}) = \left\{ M \in M_l(\mathbb{C}) \;|\; MM^\dagger \;= r\mathrm{Id}_2 \;;\; r \in \mathbb{C} \right\} \tag{0.6}$$

In the first section, we define the exponential in each basis of function $u(X)$ of matrices. Next in the second section, we give the commutation relations between generators of the Arm-Lie algebra and the u-Jacobi identity. After in the third section, we show that commutators of commutators of elements of the Arm-Lie algebras are still in the Arm-Lie algebra. This is the condition which assure that the exponential of linear combinations will be groups which we call Arm-Lie groups. Furthermore in the fourth section we give the example of the p-th root of $\mathfrak{so}(2)$ which is a trivial case because there is only two generators in the Arm-Lie algebra. Nonetheless it give rise to the new group $rSO(2)$ (0.5) and we give elements of this groups. Finally in the fifth section, we give the example of the p-th root of $\mathfrak{su}(2)$ which is not trivial because there is 6 generators in this Arm-Lie algebra. We give generating elements of its corresponding Arm-Lie group $rSU(2)$ (0.6).

1 The U-Exponential Map

Definition 1. *We call the 'u-exponential' the function*

$$e^{u(X)t} = \sum_{k=0}^{\infty} \frac{(u(X)t)^k}{k!} \tag{1.7}$$

where $u(X) \in C(M_l(\mathbb{C})), l \in \mathbb{N}$ is a function of matrices and X is a matrix.

Then we can compute the corresponding u-adjoint action

Proposition 1. *The u-adjoint action corresponding to the u-exponential is given by*

$$\mathrm{ad}_u A.X = [A, u(X)] \tag{1.8}$$

where $[\ ,\ ]$ is the traditional Lie bracket.

<u>Proof :</u>

$$\begin{aligned}
\mathrm{ad}_u A.X &= \lim_{t\to 0} \frac{d}{dt}\left[\ e^{-u(X)t}\ A\ e^{u(X)t}\ \right] \\
&= \lim_{t\to 0} \frac{d}{dt}\left[\ (1-u(X)t+...)\ A\ (1+u(X)t+...)\ \right] \\
&= \lim_{t\to 0} \frac{d}{dt}\left[\ A+t(Au(X)-u(X)A)+..\ \right] \\
&= Au(X)-u(X)A \\
\mathrm{ad}_u A.X &= [A, u(X)] \qquad (1.9)
\end{aligned}$$

♦

We call $\mathfrak{g}$ a Lie algebra.

2 The Arm-Lie Algebra

Definition 2. *A Arm-Lie algebra is a collection $\{X_1, ..., X_n, u(X_1), ..., u(X_n)\}$ such that $\{u(X_1), ..., u(X_n)\}$ is a basis of the Lie algebra $\mathfrak{g}$.*

In the Arm-Lie algebra, we have the following relations

$$\begin{aligned}
[X_i, u(X_j)] &= \lambda_{ij}{}^k u(X_k) \\
[X_i, X_j] &= \nu_{ij}{}^k u(X_k)
\end{aligned} \tag{2.10}$$

where ν and λ are the structure constant and the u-structure constant respectively.

We can also define the corresponding u-Jacobi identity

$$[u(X), [Y, Z]] + [u(Y), [Z, X]] + [u(Z), [X, Y]] = 0 \tag{2.11}$$

3 The Arm-Lie Group

First we recall the Campbell-Hausdorff formula

$$e^X e^Y = e^{Z(X,Y)} \tag{3.12}$$

where

$$Z(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X-Y,[X,Y]]) - \frac{1}{24}[Y,[X,[X,Y]]] + ... \tag{3.13}$$

The idea of the Arm-Lie group is that if we take generators of the Arm-Lie algebra $X = X_i$ and $Y = X_j$ then each terms of $Z(X,Y)$ in (3.13) would be the commutators of elements of the Arm-Lie algebra :

$$\begin{aligned}
[X_k,[...,[X_f,[X_p,X_l]]...]] &= [X_k,[...,[X_f,{\nu_{pl}}^t u(X_t)]...]] \\
&= {\nu_{pl}}^t [X_k,[...,{\lambda_{ft}}^r u(X_r)...]] \\
&\vdots \\
[X_k,[...,[X_f,[X_p,X_l]]...]] &= {\nu_{pl}}^t ... {\lambda_{km}}^s u(X_s) + ... + {\lambda_{ab}}^c ... {\lambda_{de}}^f u(X_f)
\end{aligned} \tag{3.14}$$

and the first term of $Z(X,Y)$ are given by $X_i + X_j$. Then we can conclude that the exponential of elements of an Arm-Lie algebra is a group which we naturally call Arm-Lie group.

In the moment I'm writing this article, the only Lie algebra which give Arm-Lie algebras are Lie algebra with generators which are the exponential of something (i.e. their logarithms exist). So we can start in studying the trivial one dimensional Arm-Lie algebra $\sqrt[p]{\mathfrak{so}(2)}$ and its correspondinf Arm-Lie group.

4 $\sqrt[p]{\mathfrak{so}(2)}$

We consider the changing of variable :

$$u(X) = X^p \tag{4.15}$$

for $p \in \mathbb{C}^*$. It's well known that the generator basis of $\mathfrak{so}(2)$ is given by the first Pauli matrix multiplify by $\sqrt{-1}$:

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_1) = (X_1)^p \tag{4.16}$$

Because $i\sigma_1$ is a basis of $\mathfrak{so}(2)$, if we want to know the basis of $u^{-1}(\mathfrak{so}(2)) = \sqrt[p]{\mathfrak{so}(2)}$, we have to calculate the p-th root of $i\sigma_1$. Then you can check the validity of the relation :

$$i\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \tag{4.17}$$

So with (4.17), it is very easy to calculate the p-th root of $i\sigma_1$:

$$
\begin{aligned}
\sqrt[p]{i\sigma_1} &= \exp\left(\frac{\pi}{2p}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\
&= \cos\left(\frac{\pi}{2p}\right) \mathrm{Id}_2 \; + \sin\left(\frac{\pi}{2p}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
X_1 \;=\; \sqrt[p]{i\sigma_1} &= \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix}
\end{aligned}
$$

where Id_2 is of course the identity 2-dimensional matrix. This Arm-Lie algebra is trivial because there is only 2 generators $i\sigma_1 = (X_1)^p$ and $\sqrt[p]{i\sigma_1} = X_1$ satistying the relations :

$$
\begin{aligned}
[X_1, {X_1}^p] &= 0 \\
[X_1, X_1] &= 0
\end{aligned} \tag{4.18}
$$

Of course (4.18) implies that the generator of the Arm-Lie algebra satisfy the u-Jacobi identity (2.11). Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie group :

$$
\exp(\sqrt[p]{\mathfrak{so}(2)}) = \left\{(z_1, z_2) \in \mathbb{C}^2 \;\mid\; e^{z_1 \sqrt[p]{i\sigma_1} + z_2 i\sigma_1}\right\} \tag{4.19}
$$

To have an idea of the elements of the group (4.19), we can explicit :

$$
\exp(z_1 X_1) = \exp\left(z_1 \;\; \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & -\sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \tag{4.20}
$$

and the well-known element of $SO(2)$:

$$
\exp\left(z_2 {X_1}^p\right) = \begin{pmatrix} \cos(z_2) & -\sin(z_2) \\ \sin(z_2) & \cos(z_2) \end{pmatrix} \tag{4.21}
$$

Hence we can identify this Arm-Lie algebra to

$$
\exp(\sqrt[p]{\mathfrak{so}(2)}) \equiv rSO(2, \mathbb{C}) \equiv \left\{M \in M_n(\mathbb{C}) \;\mid\; {}^tMM = r\,\mathrm{Id}_2 \;\; ; \;\; r \in \mathbb{C}\right\} \tag{4.22}
$$

for $\frac{1}{p} \neq 1[2]$ and just $SO(2)$ if $\frac{1}{p} = 1[2]$.

Remark 1. *Of course the cosinus and the sinus of a complex arument is well defined*

$$
\begin{aligned}
\cos(z) &= \cos(\mathrm{Re}(z))\cosh(\mathrm{Im}(z)) - i\sin(\mathrm{Re}z)\sinh(\mathrm{Im}(z)) \\
\sin(z) &= \sin(\mathrm{Re}(z))\cosh(\mathrm{Im}(z)) + i\cos(\mathrm{Re}(z))\sinh(\mathrm{Im}(z))
\end{aligned} \tag{4.23}
$$

where Re and Im denote the real and imaginary parts respectively.

5 $\sqrt[p]{\mathfrak{su}(2)}$

We still consider the changing of variable :

$$u(X) = X^p \tag{5.24}$$

for $p \in \mathbb{C}^*$. It's well known that the generator basis of $\mathfrak{su}(2)$ is given by the Pauli matrices multiplify by $\sqrt{-1}$:

$$\begin{aligned}
i\sigma_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u(X_1) = (X_1)^p \\
i\sigma_2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = u(X_2) = (X_2)^p \\
i\sigma_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = u(X_3) = (X_3)^p
\end{aligned} \tag{5.25}$$

Because $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ is a basis of $\mathfrak{su}(2)$, if we want to know the basis of $u^{-1}(\mathfrak{so}(2)) = \sqrt[p]{\mathfrak{su}(2)}$, we have to calculate the p-th root of $i\sigma_1, i\sigma_2$ and $i\sigma_3$. Then you can check the validity of the relation :

$$\begin{aligned}
i\sigma_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\
i\sigma_2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) \\
i\sigma_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp\left(\frac{\pi}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right)
\end{aligned}$$

So with (5.26), it is very easy to calculate the p-th root of $i\sigma_1, i\sigma_2, i\sigma_3$:

$$\begin{aligned}
& & \sqrt[p]{i\sigma_1} &= \exp\left(\frac{\pi}{2p}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\
& & &= \cos\left(\frac{\pi}{2p}\right)\ \mathrm{Id}_2\ + \sin\left(\frac{\pi}{2p}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
X_1 &= & \sqrt[p]{i\sigma_1} &= \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & -\sin\left(\frac{\pi}{2p}\right) \\ \sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix} \\
& & \sqrt[p]{i\sigma_2} &= \exp\left(\frac{\pi}{2p}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right) \\
& & &= \cos\left(\frac{\pi}{2p}\right)\ \mathrm{Id}_2\ + \sin\left(\frac{\pi}{2p}\right)\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
X_2 &= & \sqrt[p]{i\sigma_2} &= \begin{pmatrix} \cos\left(\frac{\pi}{2p}\right) & i\sin\left(\frac{\pi}{2p}\right) \\ i\sin\left(\frac{\pi}{2p}\right) & \cos\left(\frac{\pi}{2p}\right) \end{pmatrix} \\
& & \sqrt[p]{i\sigma_3} &= \exp\left(\frac{\pi}{2p}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right) \\
X_3 &= & \sqrt[p]{i\sigma_3} &= \begin{pmatrix} \exp\left(\frac{\pi}{2p}\right) & 0 \\ 0 & \exp\left(-\frac{\pi}{2p}\right) \end{pmatrix}
\end{aligned}$$

where Id_2 is of course the identity 2-dimensional matrix. This Arm-Lie algebra has 6 generators $i\sigma_1 = (X_1)^p, i\sigma_2 = (X_2)^p, i\sigma_3 = (X_3)^p$ and $\sqrt[p]{i\sigma_1} = X_1$, $\sqrt[p]{i\sigma_2} = X_2$, $\sqrt[p]{i\sigma_3} = X_3$ satistying the relations :

$$\begin{aligned}
[X_1, (X_2)^p] &= -2\sin\left(\frac{\pi}{2p}\right)\ (X_3)^p \\
[X_2, (X_3)^p] &= -2\sin\left(\frac{\pi}{2p}\right)\ (X_1)^p \\
[X_3, (X_1)^p] &= -2\sin\left(\frac{\pi}{2p}\right)\ (X_2)^p
\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}
[X_1, X_2] &= -2\sin^2\left(\frac{\pi}{2p}\right)\ (X_3)^p \\
[X_2, X_3] &= -2\sin^2\left(\frac{\pi}{2p}\right)\ (X_1)^p \\
[X_3, X_1] &= -2\sin^2\left(\frac{\pi}{2p}\right)\ (X_2)^p
\end{aligned} \tag{5.27}$$

Of course (5.27) and (5.26) imply that the generators of the Arm-Lie algebra $\sqrt[p]{su(2)}$ satisfy the u-Jacobi identity. Even if this Arm-Lie algebra is trivial it give rise to an interesting new Arm-Lie group :

$$\exp(\sqrt[p]{\mathfrak{su}(2)}) = \left\{(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{C}^6 \mid e^{z_1 \sqrt[p]{i\sigma_1} + z_2 i\sigma_1 + z_3 \sqrt[p]{i\sigma_2} + z_4 i\sigma_2 + z_5 \sqrt[p]{i\sigma_3} + z_6 i\sigma_3}\right\} \tag{5.28}$$

To have an idea of the elements of the group (4.19), we can explicit :

$$\begin{aligned}
\exp(z_1 X_1) &= \exp\left(z_1 \ \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & -\sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \\ \sin\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_1 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \\
\exp(z_3 X_3) &= \exp\left(z_3 \ \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \cos\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) & i\sin\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) \\ i\sin\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) & \cos\left(z_3 \sin\left(\frac{\pi}{2p}\right)\right) \end{pmatrix} \\
\exp(z_5 X_5) &= \exp\left(z_5 \ \cos\left(\frac{\pi}{2p}\right)\right) \begin{pmatrix} \exp\left(iz_5 \sin\left(\frac{\pi}{2p}\right)\right) & 0 \\ 0 & \exp\left(z_5 \sin\left(-\frac{\pi}{2p}\right)\right) \end{pmatrix}
\end{aligned} \tag{5.29}$$

and the well-known elements of $SU(2)$:

$$\begin{aligned}
\exp\left(z_2 {X_1}^p\right) &= \begin{pmatrix} \cos(z_2) & -\sin(z_2) \\ \sin(z_2) & \cos(z_2) \end{pmatrix} \\
\exp\left(z_4 {X_2}^p\right) &= \begin{pmatrix} \cos(z_4) & i\sin(z_4) \\ i\sin(z_4) & \cos(z_4) \end{pmatrix} \\
\exp\left(z_6 {X_3}^p\right) &= \begin{pmatrix} \exp(iz_6) & 0 \\ 0 & \cos(-iz_6) \end{pmatrix}
\end{aligned} \tag{5.30}$$

Hence we can identify this Arm-Lie algebra to

$$\exp(\sqrt[p]{\mathfrak{su}(2)}) \equiv rSU(2,\mathbb{C}) \equiv \left\{M \in M_n(\mathbb{C}) \mid M^\dagger M = r \,\mathrm{Id}_2 \ ; \ r \in \mathbb{C}\right\} \tag{5.31}$$

for $\frac{1}{p} \neq 1[2]$ and just $SU(2)$ if $\frac{1}{p} = 1[2]$.

Discussion

Unfortunately, I searched other Arm-Lie algebras among classics Lie algebras but I didn't find. I think it's because many other classic groups are not the exponential of something (i.e. their logarithms do not exist) but I'm not sure. I find this structure only for $\mathfrak{su}(2)$ and its subalgebra $\mathfrak{so}(2)$. I tried with $\mathfrak{so}(3)$ and $\mathfrak{sl}(2)$ but it didn't work. I develop this theory in order to classify what I found but I hope there is other Lie algebras which are Arm-Lie algebras. There is not a lot of example of Arm-Lie algebra but $\sqrt[p]{\mathfrak{su}(2)}$ is a beautiful and very fondamental example.

I also tried this theory for other $u(X)$ for example I took $u(X)$ =exp(exp(X)) but it is just a shifted case of the traditional case. I was also limited by the choice of changing of variable that I was able to do because I had to find the inverse function of matricial variable which is sometimes hard to do. I also tried function as sin or cos but it gave the identity or zero respectively. The only good changing of variable which I found was $u(X) = X^p$ which is a lot of changing for $p \in \mathbb{C}$ but it finally give the same result for all p.

Finally we can imagine an other Lie group theory based on the 'p-exponential' which I introduced in [2] but now I think that I will give the same result as the usual Lie group theory but with the p-exponential instead of the traditional exponential. Maybe I will explore this way in an other work.

Références

[1] Arm B. N., The Arm Theory

[2] Arm B. N., The p-Arm Theory