Some Poisson-Lie sigma models

ARM Boris

arm.boris@gmail.com

Abstract

We calculate the Poisson-Lie sigma model for every 4-dimensional Manin triples (function of its structure constant) and we give the 6-dimensional models for the Manin triples $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C})^*)$, $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C})^*), \mathfrak{sl}(2,\mathbb{C})$, $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}))$ and $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}))$

1 Introduction

A Manin triples $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is a bialgebra $(\mathfrak{g}, \tilde{\mathfrak{g}}$ which don't intersect each others and a direct sum of this bialgebra $\mathfrak{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}})$. If the corresponding Lie groups have a Poisson structure, they are called Poisson-Lie groups. A Poisson-Lie sigma models is an action (3.13) calculated by a Poisson vector field matrix. [3] have deduced the extremal field which minimize the action of this models, which gives the motion equation (3.19). We calculate here the action and the equations of motion for some 6-dimensionals Manin triples and we give a general formula for each 4-dimensional Manin triples. The 6-dimensional Manin triples are $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C})^*)$,$(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C})^*), \mathfrak{sl}(2,\mathbb{C})$,$(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}))$ and $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}))$.

2 Some Manin triples

The Drinfeld double D is defined as a Lie group such that its Lie algebra $\mathfrak{D}$ equipped by a symmetric ad-invariant nondegenerate bilinear form $< .,. >$ can be decomposed into a pair of maximally isotropic subalgebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ such that $\mathfrak{D}$ as a vector space is the direct sum of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. Any such decomposition written as an ordered set $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$ is called a Manin triples $(\mathfrak{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$,$(\mathfrak{D}, \tilde{\mathfrak{g}}, \mathfrak{g})$.

One can see that the dimensions of the subalgebras are equal and that bases $\{T_i\}, \{\tilde{T^i}\}$ in the subalgebras can be chosen so that

$$< T_i, T_j >= 0, \quad < T_i, \tilde{T}^j >=< \tilde{T}^j, T_i >= \delta_i^j, \quad < \tilde{T}^i, \tilde{T}^j >= 0 \tag{2.1}$$

This canonical form of the bracket is invariant with respect to the transformations

$$T'_i = T_k A_i^k, \quad \tilde{T}'^j = (A^{-1})_k^j \tilde{T}^k \tag{2.2}$$

Due to the ad-invariance of $< .,. >$ the algebraic structure of $\mathfrak{D}$ is

$$[T_i, T_j] = c_{ij}{}^k T_k, \qquad [\tilde{T}^i, \tilde{T}^j] = f^{ij}{}_k \tilde{T}^k$$
$$[T_i, \tilde{T}^j] \;=\; f^{jk}{}_i T_k - c_{ik}{}^j \tilde{T}^k$$

There are just four types of nonisomorphic four-dimensional Manin triples.
Abelian Manin triples :

$$[T_i, T_j] = 0, \qquad [\tilde{T}^i, \tilde{T}^j] = 0, \qquad [T_i, \tilde{T}^j] = 0, \qquad i, j = 1, 2 \tag{2.3}$$

Semi-Abelian Manin triples (only non trivial brackets are displayed) :

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \qquad [T_2, \tilde{T}^1] = T_2, \qquad [T_2, \tilde{T}^2] = -T_1 \tag{2.4}$$

Type A non-Abelian Manin triples ($\beta \neq 0$) :

$$[T_1, T_2] = T_2, \qquad [\tilde{T}^1, \tilde{T}^2] = \beta \tilde{T}^2$$
$$[T_1, \tilde{T}^2] = -\tilde{T}^2, \qquad [T_2, \tilde{T}^1] = \beta T_2, \qquad [T_2, \tilde{T}^2] = -\beta T_1 + \tilde{T}^1$$

Type B non-Abelian Manin triples :

$$[T_1, T_2] = T_2, \qquad [\tilde{T}^1, \tilde{T}^2] = \tilde{T}^1$$
$$[T_1, \tilde{T}^1] = T_2, \qquad [T_1, \tilde{T}^2] = -T_1 - \tilde{T}^2, \qquad [T_2, \tilde{T}^2] = \tilde{T}^1$$

Now we focus some six dimensional Manin triples. We recall that the commutation relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of the Lie group $SL(2, \mathbb{C})$:

$$[T_1, T_2] = 2T_2, \quad [T_1, T_3] = -2T_3, \quad [T_2, T_3] = T_1 \tag{2.5}$$

The dual Lie algebra $\mathfrak{sl}(2, \mathbb{C})^*$ of the Lie algebra $\mathfrak{sl}(2, C)$ has the commutation relations :

$$[\tilde{T}^1, \tilde{T}^2] = \frac{1}{4}\tilde{T}^2, \quad [\tilde{T}^1, \tilde{T}^3] = \frac{1}{4}\tilde{T}^3, \quad [\tilde{T}^2, \tilde{T}^3] = 0 \tag{2.6}$$

There is a scalar product on $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*$ such that (see [2]) :

$$(T_i, \tilde{T}^j) = \delta_i^j \tag{2.7}$$

Finally, we have that $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C})^*)$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C}))$ with this scalar product is also a Manin triples.

The Iwasawa decomposition allows us to decompose :

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{sb}(2,\mathbb{C}) \tag{2.8}$$

where $\mathfrak{su}(2,\mathbb{C})$ is the Lie algebra of the Lie group $SU(2)$ with commutation relations :

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2 \tag{2.9}$$

$\mathfrak{sb}(2,\mathbb{C})$ is the Lie algebra of the Borel subgroup $SB(2,\mathbb{C})$ with commutation relations :

$$[\tilde{T}^1, \tilde{T}^2] = \tilde{T}^2, \quad [\tilde{T}^1, \tilde{T}^3] = \tilde{T}^3, \quad [\tilde{T}^2, \tilde{T}^3] = 0 \tag{2.10}$$

Here we can see in comparing (2.10) and (2.6) that $\mathfrak{sb}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})^*$.

The Iwasawa decomposition (2.8) allows us to identify $\mathfrak{sb}(2,\mathbb{C}) \simeq \mathfrak{su}(2,\mathbb{C})^*$. We define a scalar product on $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*$ such that $(x,y) =$Im(Tr$(x|y)$). With this scalar product we have (see [2]) :

$$(T_i, \tilde{T}^j) = \delta_i^j \tag{2.11}$$

Finally we have that $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}))$ with this scalar product is a Manin triple. We note that $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}))$ with this scalar product is also a Manin triples.

3 Poisson-Lie sigma models

Given a Lie group M and a Poisson structure on it. We define the action of this model (see [1]) as :

$$S_1 = \int_\Sigma (< dgg^{-1}, A > -\frac{1}{2} < A, (r - Ad_g r Ad_g) A >) \tag{3.12}$$

where $g \in G, A = A^i_\alpha d\xi^\alpha X_i$ and $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a classical r matrix with $\mathfrak{g}$ as the Lie algebra of G and $\{X_i\}$ as a basis of $\mathfrak{g}$. Note that the above action can be applied for simple or nonsemisimple Lie group G with ad-invariant symmetric bilinear nondegenerate form $< X_i, X_j >= G_{ij}$ on the Lie algebra $\mathfrak{g}$. When the metric G_{ij} of Lie algebra is denegerate then the above action is not good. Here we use the following action instead of the above one :

$$S_2 = \int_\Sigma (dX_i \wedge A_i - \frac{1}{2}\mathcal{P}^{ij} A_i \wedge A_j) \tag{3.13}$$

where x are Lie group parameters with parametrization (e.g.)

$$\forall g \in G, g = e^{X_1 T_1} e^{X_2 T_2} ... \tag{3.14}$$

where P^{ij} is the Poisson structure on the Lie group which for coboundary Poisson Lie groups it is obtained from

$$(\mathcal{P}(g))_\chi = b(g)a(g)^{-1} \tag{3.15}$$

We can obtain $a(g)^{-1}$ and $b(g)$ in computing :

$$(Ad_{g^{-1}})_\chi = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix} \tag{3.16}$$

$$(Ad_g)_\chi = \begin{pmatrix} a(g)^{-T} & -a(g)^{-T}b(g)^T d(g)^{-T} \\ 0 & d(g)^{-T} \end{pmatrix} \tag{3.17}$$

where ^{T} denotes the transpose matrix.

The extremal fiels (X, A) which minimize the action (3.13) have to satisfy the equation written locally (see [3]) as :

$$dX_i + \mathcal{P}^{ij}(X)A_j = 0 \tag{3.18}$$

$$dA_k + \frac{1}{2}\mathcal{P}^{ij}{}_{,k}(X)A_i \wedge A_j = 0 \tag{3.19}$$

where $\mathcal{P}^{ij}{}_{,k} = \partial_k \mathcal{P}^{ij}|_{X_k=0}$.

4 Poisson-Lie sigma model of any 4-dimensional Manin triple

We first calculate the matrix of the adjoint actions function of structure constant :

$$ad_{T_1} = \begin{pmatrix} 0 & c_{12}{}^1 & 0 & -f^{12}{}_1 \\ 0 & c_{12}{}^2 & f^{12}{}_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c_{12}{}^1 & -c_{12}{}^2 \end{pmatrix}$$

$$ad_{T_2} = \begin{pmatrix} -c_{12}{}^1 & 0 & 0 & -f^{12}{}_2 \\ -c_{12}{}^2 & 0 & f^{12}{}_2 & 0 \\ 0 & 0 & c_{12}{}^1 & c_{12}{}^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To obtain the matrix $\mathcal{P}$, we calculate the adjoint action matrix of a general element $g = \prod_{i=1}^2 e^{\alpha_i T_i}$ by the formula :

$$(Ad_{\prod_{i=1}^2 e^{X_i T_i}})_\chi = \prod_{i=1}^2 e^{X_i (ad_{T_i})_\chi} \tag{4.20}$$

Similarily, we have :

$$(Ad_{(\prod_{i=1}^2 e^{X_i T_i})^{-1}})_\chi = \prod_{i=1}^2 e^{-X_{3-i}(ad_{T_{3-i}})_\chi} \tag{4.21}$$

We can deduce the matrix $\mathcal{P}^{ij}$:

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -\mathcal{P}^{21} \\ \mathcal{P}^{21} & 0 \end{pmatrix}$$

where

$$\mathcal{P}^{21} = \frac{{c_{12}}^1(-1+e^{{c_{12}}^2 X_1}){f^{12}}_1 + {c_{12}}^2 e^{{c_{12}}^2 X_1 - {c_{12}}^1 X_2}(-1+e^{{c_{12}}^1 X_2}){f^{12}}_2}{{c_{12}}^2 {c_{12}}^1} \tag{4.22}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_\Sigma \sum_{i=1}^{2} dX_i \wedge A_i - \mathcal{P}^{21} A_2 \wedge A_1 \tag{4.23}$$

and the equations of motion (3.19) :

$$\begin{aligned}
dX_1 - \mathcal{P}^{21} A_2 &= 0 \\
dX_2 + \mathcal{P}^{21} A_1 &= 0 \\
dA_1 - \frac{{c_{12}}^1 {c_{12}}^2 {f^{12}}_1 + {c_{12}}^2 {c_{12}}^2(-e^{-{c_{12}}^1 X_2}+1){f^{12}}_2}{{c_{12}}^2 {c_{12}}^1} A_1 \wedge A_2 &= 0 \\
dA_2 + \frac{{c_{12}}^2 e^{{c_{12}}^2 X_1} {c_{12}}^1 {f^{12}}_2}{{c_{12}}^2 {c_{12}}^1} A_2 \wedge A_1 &= 0
\end{aligned} \tag{4.24}$$

5 Poisson-Lie sigma model of $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{C})^*)$

We first calculate the matrix of the adjoint actions :

$$\begin{aligned}
ad_{T_1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \\
ad_{T_2} &= \frac{1}{4} \begin{pmatrix} 0 & 0 & 4 & 0 & -1 & 0 \\ -8 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \end{pmatrix} \\
ad_{T_3} &= \frac{1}{4} \begin{pmatrix} 0 & -4 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

To obtain the matrix $\mathcal{P}$, we calculate the adjoint action matrix of a general element $g = \prod_{i=1}^{3} e^{\alpha_i T_i}$ by the formula :

$$(Ad_{\prod_{i=1}^{3} e^{X_i T_i}})_\chi = \prod_{i=1}^{3} e^{X_i (ad_{T_i})_\chi} \tag{5.25}$$

Similarily, we have :

$$(Ad_{(\prod_{i=1}^{3} e^{X_i T_i})^{-1}})_\chi = \prod_{i=1}^{3} e^{-X_{4-i} (ad_{T_{4-i}})_\chi} \tag{5.26}$$

We can deduce the matrix $\mathcal{P}^{ij}$:

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -\frac{X_2}{4}(1+X_2X_3)e^{2X_1} & -\frac{X_3}{4}e^{-2X_1} \\ \frac{X_2}{4}(1+X_2X_3)e^{2X_1} & 0 & \frac{X_2X_3}{2} \\ \frac{X_3}{4}e^{-2X_1} & -\frac{X_2X_3}{2} & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_\Sigma \sum_{i=1}^{3} dX_i \wedge A_i + (\frac{X_2}{4}(1+X_2X_3)e^{2X_1})A_1 \wedge A_2 + \frac{X_3}{4}e^{-2X_1} A_1 \wedge A_3 - \frac{X_2X_3}{2} A_2 \wedge A_3 \tag{5.27}$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 - (\frac{X_2}{4}(1+X_2X_3)e^{2X_1})A_2 - \frac{X_3}{4}e^{-2X_1}A_3 = 0 \\ dX_2 + (\frac{X_2}{4}(1+X_2X_3)e^{2X_1})A_1 + \frac{X_2X_3}{2}A_3 = 0 \\ dX_3 + \frac{X_3}{4}e^{-2X_1}A_1 - \frac{X_2X_3}{2}A_2 = 0 \\ dA_1 - \frac{X_2}{2}(1+X_2X_3)A_1 \wedge A_2 + \frac{X_3}{2}A_1 \wedge A_2 = 0 \\ dA_2 - \frac{e^{2X_1}}{4}A_1 \wedge A_2 + \frac{X_3}{2}A_2 \wedge A_3 = 0 \\ dA_3 - \frac{X_2^2}{4}e^{2X_1}A_1 \wedge A_2 - \frac{e^{-2X_1}}{4}A_1 \wedge A_3 + X_2 A_2 \wedge A_3 = 0 \end{aligned}$$

6 Poisson-Lie sigma model of $(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C})^*, \mathfrak{sl}(2,\mathbb{C}))$

Now to obtain this Poisson Lie sigma model, we have to change $T_i \to \tilde{T}^i$ and $\tilde{T}^i \to T_i$ of the previous model. And we can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix $\mathcal{P}^{ij}$ for this model :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -2e^{\frac{X_1}{4}}X_2 & 2e^{\frac{X_1}{4}}X_3 \\ 2e^{\frac{X_1}{4}}X_2 & 0 & 2-\frac{1}{2}e^{\frac{X_1}{2}}(4+X_2X_3) \\ -2e^{\frac{X_1}{4}}X_3 & -2+\frac{1}{2}e^{\frac{X_1}{2}}(4+X_2X_3) & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_\Sigma \sum_{i=1}^{3} dX_i \wedge A_i + 2e^{\frac{X_1}{4}} X_2 A_1 \wedge A_2 - 2e^{\frac{X_1}{4}} X_3 A_1 \wedge A_3 + (-2 + \frac{1}{2} e^{\frac{X_1}{2}} (4 + X_2 X_3)) A_2 \wedge A_3 \quad (6.28)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 - 2e^{\frac{X_1}{4}} X_2 A_2 + 2e^{\frac{X_1}{4}} X_3 A_3 = 0 \\ dX_2 + 2e^{\frac{X_1}{4}} X_2 A_1 + (2 - \frac{1}{2} e^{\frac{X_1}{2}} (4 + X_2 X_3)) A_3 = 0 \\ dX_3 - 2e^{\frac{X_1}{4}} X_3 A_1 - (2 - \frac{1}{2} e^{\frac{X_1}{2}} (4 + X_2 X_3)) A_2 = 0 \\ dA_1 - \frac{X_2}{2} A_1 \wedge A_2 + \frac{X_3}{4} A_1 \wedge A_3 - \frac{1}{4}(4 + X_2 X3) A_2 \wedge A_3 = 0 \\ dA_2 - 2e^{\frac{X_1}{4}} A_1 \wedge A_2 - \frac{1}{2} e^{\frac{X_1}{4}} A_2 \wedge A_3 = 0 \\ dA_3 + 2e^{\frac{X_1}{4}} A_1 \wedge A_3 - \frac{e^{\frac{X_1}{2}} X_2}{2} A_2 \wedge A_3 = 0 \end{aligned}$$

7 Poisson-Lie sigma model of $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}))$

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix $\mathcal{P}^{ij}$ for this model :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3 & -\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3 \\ \cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3 & 0 & -1 + \cos X_2 \cos X_3 \\ \cos X_3 \sin X_1 \sin X_3 + \cos X_1 \sin X_3 & 1 - \cos X_2 \cos X_3 & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$\begin{aligned} S_2 = \int_\Sigma \sum_{i=1}^{3} dX_i \wedge A_i - (-\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3) A_1 \wedge A_2 \\ -(-\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3) A_1 \wedge A_3 - (-1 + \cos X_2 \cos X_3) A_2 \wedge A_3 \end{aligned} \quad (7.29)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 + (-\cos X_1 \cos X_3 \sin X_2 + \sin X_1 \sin X_3) A_2 + (-\cos X_3 \sin X_1 \sin X_2 - \cos X_1 \sin X_3) A_3 = 0 \\ dX_2 + (\cos X_1 \cos X_3 \sin X_2 - \sin X_1 \sin X_3) A_1 + (-1 + \cos X_2 \cos X_3) A_3 = 0 \\ dX_3 + (\cos X_3 \sin X_1 \sin X_3 + \cos X_1 \sin X_3) A_1 (1 - \cos X_2 \cos X_3) A_2 = 0 \\ dA_1 + \sin X_3 A_1 \wedge A_2 - \cos X_3 \sin X_2 A_1 \wedge A_3 = 0 \\ dA_2 - \cos X_1 \cos X_3 A_1 \wedge A_2 - \cos X_3 \sin X_1 A_1 \wedge A_3 = 0 \\ dA_3 + \sin X_1 A_1 \wedge A_2 - \cos X_1 A_1 \wedge A_3 = 0 \end{aligned}$$

8 Poisson-Lie sigma model of $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{sb}(2,\mathbb{C}), \mathfrak{su}(2,\mathbb{C}))$

We can calculate the matrix of the adjoint actions as we do previously. With this we can deduce the matrix $\mathcal{P}^{ij}$ for this model :

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -e^{X_1}X_3 & -e^{X_1}X_2 \\ e^{X_1}X_3 & 0 & \frac{1}{2}(1-e^{2X_1}(1+X_2^2+X_3^2)) \\ e^{X_1}X_2 & \frac{1}{2}(-1+e^{2X_1}(1+X_2^2+X_3^2)) & 0 \end{pmatrix}$$

Now, we can calculate the action (3.13) of the model

$$S_2 = \int_\Sigma \sum_{i=1}^{3} dX_i \wedge A_i + e^{X_1}X_3 A_1 \wedge A_2 + e^{X_1}X_2 A_1 \wedge A_3 - \frac{1}{2}(1-e^{2X_1}(1+2X_2^2+2X_3^2))A_2 \wedge A_3 \quad (8.30)$$

and the equations of motion (3.19) :

$$\begin{aligned} dX_1 - e^{X_1}X_3A_2 - e^{X_1}X_2A_3 &= 0 \\ dX_2 + e^{X_1}X_3A_1 + \frac{1}{2}(1-e^{2X_1}(1+X_2^2+X_3^2))A_3 &= 0 \\ dX_3 + e^{X_1}X_2A_1 - \frac{1}{2}(1-e^{2X_1}(1+X_2^2+X_3^2))A_2 &= 0 \\ dA_1 - X_3A_1\wedge A_2 - X_2A_1\wedge A_3 - (1+X_2^2+X_3^2)A_2\wedge A_3 &= 0 \\ dA_2 - e^{X_1}A_1\wedge A_3 &= 0 \\ dA_3 - e^{X_1}A_1\wedge A_2 &= 0 \end{aligned}$$

9 Discussion

We gives here the Poisson-Lie sigma models of some Manin triples. Concerning the general formula (9.32), we have to say that this is no problem when ${c_{12}}^2$ and ${c_{12}}^2$ is zero because

$$\mathcal{P}^{21} = \frac{(-1+e^{{c_{12}}^2X_1}){f^{12}}_1}{{c_{12}}^2} + \frac{e^{{c_{12}}^2X_1-{c_{12}}^1X_2}(-1+e^{{c_{12}}^1X_2}){f^{12}}_2}{{c_{12}}^2} \quad (9.31)$$

which can be approximate by

$$\mathcal{P}^{21} = (X_1 + \frac{{c_{12}}^2}{2}X_1^2 + ...){f^{12}}_1 + e^{{c_{12}}^2X_1-{c_{12}}^1X_2}(X_2 + \frac{{c_{12}}^1}{2}X_2^2 + ...){f^{12}}_2 \quad (9.32)$$

We tried to obtain the equivalent formula for $n = 3$ but the calculus was too hard.

Références

[1] Hajizadeh S., Rezaei-Aghdam A., Poisson-Lie Sigma models over low dimensional real Poisson-Lie groups

[2] Kosmann-Schwarzbach Y., Lie bialgebras, Poisson Lie Groups and Dressing Transformation

[3] Vysoký J., Hlavatý , Poisson Lie Sigma Models on Drinfeld double