A Mathematical Analysis of Crowds

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Abstract

Crowds are generally analyzed in the regime of sociology- where they are studied and classified on the basis of crowd psychology. This analysis arises from the study of collective behavior and treats crowds as dependent on psychology of humans in the crowd. In this introductory paper we show a generalized treatment of crowds as a set of living objects: called members of the crowd. We classify crowds based on various parameters and study some general and specific characteristics of crowd of humans and study the response of a simple crowd to an external situation or stimulus by deriving the solution of the generalized crowd equation. We also define some terminology regarding the mathematical description of crowds and hence arrive at some useful conjectures.

1.Crowd

A crowd is defined in general as a set of n-living objects (called members of the crowd) in a given region of space-time such that each member has a response potential(S) associated with it. For a given member, the response potential is a scalar that quantifies the effect of a given situation on that member.

In the most general form, a crowd is written as a finite set

$$C=\{M_1[S_1], M_2[S_2], \ldots., M_n[S_n]\}$$

where S is the response potential for a given member. The response potential is the value of a more general multivariate function called the Response Function and is given as

$s(D,r)=|S|$ and for the ith member, $s(D,r)=|S|=S_i$

Here, 'D' is called the Defining Parameter and 'r' is called the Separation Parameter.

Conjecture 1: The response Potential is directly proportional to the change in position value of a given member w.r.t time. Mathematically,

$$S_i \propto \frac{dP_i}{dt}$$

Hence,

$$S_i = K\left(\frac{dP_i}{dt}\right) \text{ this is the Crowd Equation}$$

The Crowd Equation is a first order Ordinary Differential Equation. The constant of proportionality K is called the Crowd Inertia and is constant for a given crowd and its value is same for all members of crowd (this assumption holds for ideal crowds).
A more general form of the crowd equation can be obtained if we have

$$D = f(x, y, , , ,) = f(\eta)$$

This implies

$$s(D, r) = s(f(\eta), r)$$

Hence,

$$s(f(\eta), r) = K\left(\frac{dP_i}{dt}\right) \text{ this is the generalized crowd equation}$$

The above equation applies to any situation for which a Response Function can be written.
The crowd inertia is defined as "the property of a crowd by virtue of which it resists any change in the position of its members w.r.t time".

Conjecture 2: A crowd with a finite value of crowd inertia is affected by an external stimulus or situation that changes the position of its members according to a crowd equation derived from the generalized crowd equation.

We can write $$\frac{dP_i}{dt} = \frac{S_i}{K}$$

which implies that for a given situation corresponding to the response function s, a higher value of crowd inertia directly means that the position response (change in position of members due to the situation) of the crowd is low. In mathematical terms,

$$K = \frac{S_i}{\left(\frac{dP_i}{dt}\right)}$$

2.Classification of Crowds

To allow ease of study and analysis, a proper classification of crowds is required. We classify crowd on the basis of parameters defined by the variables in the crowd equation.

Static Crowd: A crowd in which the position of its members do not change with time is called static crowd. Example- audience
For a static crowd,

$$\frac{dP_i}{dt} = 0$$

Therefore $S_i = 0$

Also, a crowd is static if $K = \infty$ which is the condition for a rigid crowd which is a type of static crowd.
That is, rigid crowds are characterized by an infinite crowd inertia.

Dynamic Crowd: A crowd in which the position of its members change with time is called a dynamic crowd. Example- pedestrians on a street
For a dynamic crowd,

$$\frac{dP_i}{dt} \neq 0$$

Single-level Crowd: A crowd extended in 2 or less spatial dimensions is called a single-level crowd.
Example- crowd gathered on a plane, a queue (one dimensional crowd)

Multilevel Crowd: A crowd extended in more that 2 dimensions are called multilevel crowd. Example- crowd in a whole building: on different floors.

Conjecture 3: Any multilevel crowd can be expressed as a collection of single-level crowds.

Mob (aggressive crowd): A crowd which has a large value of response potential and crowd inertia is called an aggressive crowd or Mob.

$S_i \gg 0 \wedge K \gg 0$ This implies $\frac{dP_i}{dt}$ has a low value close to one.

3.Characteristics of a Crowd

A close observation of human crowds: both physically and mathematically reveals some of their peculiar features. We briefly discuss them here.

(a) Existence of sub-crowds

We have defined a crowd C as a set of members with each member having a definite response potential. Now,
any finite subset of C is called a sub-crowd.
i.e. $A \subset C$ defines a sub-crowd

(b) Formation of Queues

$\forall c_1, c_2, \ldots.., c_n \in C$ we define a Queuing Set Q as

$$Q = \{c_1, c_{2,} \ldots.., c_n\}$$

If T is a relation such that $c_a T c_b$ implies that a queue is formed between c_a and c_b. We can write

$$c_a T c_b = T(c_a, c_b)$$

here T is called a Queuing Relation.
The complete Queuing Relation between 4 members(say) is thus

$$T = \{(c_a, c_b), (c_b, c_c), (c_c, c_d)\}$$

We define a Queue as a discrete structure $q(Q, T)$

It is evident from sections (a) and (b) that a Queuing Set can be treated as a sub-crowd and a queue 'q' is an ordered sub-crowd. We hence arrive at our 4th conjecture.

Conjecture 4: A queue is an ordered subset of a crowd.

It follows that a Queuing Set and a Sub-Crowd are essentially equivalent i.e.

$$Q \equiv A$$

This is called the **General Crowd Equivalence.**
(c) Effect of a situation on a crowd
In our current terminology**,** a situation maybe defined as something with a definite position that can affect the crowd generally by altering the position of its members. The existence of a situation is defined by a non zero value of the response function(s).
A situation can alter the crowd in a large variety of ways as can be deduced from the crowd equation.

Conjecture 5: A response function is uniquely determined for unique values of the Defining Parameter D and the Separation Parameter r

Conjecture 5 implies that the uniqueness of situation depends upon the values of D and r. The response function is a function of D and r. D is also a multivariate function as mentioned earlier.
The separation parameter for a particular member determine the proximity of the situation to that member. Clearly, for the i^{th} member,

$$r_i = r_S - P_i$$

where the terms in the RHS denote the positions of the situation and the member respectively.

By writing different forms of the response function in terms of D and r, we can calculate (after solving the crowd equation for the given response function) the change in position of members of the crowd.

In this paper, we consider the most basic solution of the crowd equation where

$$D \equiv D(t) \quad \text{and} \quad r \equiv r(t)$$

The crowd equation now becomes hugely simplified,

$$s(D(t), r(t)) = K\left(\frac{dP_i}{dt}\right) \quad \text{(equation for a Simple Crowd)}$$

Hence, the solution is

$$P_i = \frac{1}{k}\left(\int s(D(t), r(t))\, dt\right)$$

The Crowd Equation can be solved for more complicated situations for which the integral on the RHS may extend to multiple spatial dimensions and the solution of which may require numerical methods.