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A new Smarandache function and its elementary properties

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Abstract For any positive integer n, we define a new Smarandache function $G(n)$ as the smallest positive integer m such that $\prod_{k=1}^{m} \phi(k)$ is divisible by n, where $\phi(n)$ is the Euler function. The main purpose of this paper is using the elementary methods to study the elementary properties of $G(n)$, and give three interesting formulas for it.

Keywords A new Smarandache function, series, inequality.

§1. Introduction and results

For any positive integer n, we define a new Smarandache function $G(n)$ as the smallest positive integer m such that $\prod_{k=1}^{m} \phi(k)$ is divisible by n. That is,

$$G(n) = \min\{m: \ n \mid \prod_{k=1}^{m} \phi(k), \ m \in N\},$$

where $\phi(n)$ is the Euler function, N denotes the set of all positive integers. For example, the first few values of $G(n)$ are: $G(3) = 7$, $G(4) = 4$, $G(5) = 11$, $G(6) = 7$, $G(7) = 29$, $G(8) = 5$, $G(9) = 9$, $G(10) = 11$, $G(11) = 23$, $G(12) = 7$, $G(13) = 43$, $G(14) = 29$, $G(15) = 11$, $G(16) = 5$ $\cdots$. About the properties of this function, it seems that none had studied it yet, at last we have not seen any related papers before. Recently, Professor Zhang Wenpeng asked us to study the arithmetical properties of $G(n)$. The main purpose of this paper is using the elementary methods to study this problem, and prove the following three conclusions:

Theorem 1. For any prime p, we have the calculating formulae

$$G(p) = \min\{p^2, \ q(p, \ 1)\};$$

$$G(p^2) = q(p, \ 2), \text{ if } q(p, \ 2) < p^2; \ G(p^2) = p^2, \text{ if } q(p, \ 1) < p^2 < q(p, \ 2);$$

$$G(p^2) = q(p, \ 1), \text{ if } p^2 < q(p, \ 1) < 2p^2; \text{ and } G(p^2) = 2p^2, \text{ if } q(p, \ 1) > 2p^2,$$

where $q(p, \ i)$ is the i-th prime in the arithmetical series $\{np + 1\}$.

Theorem 2. $G(n)$ is a Smarandache multiplicative function, and moreover, the Dirichlet series $\sum_{n=1}^{\infty} \frac{G(n)}{n^2}$ is divergent.

Theorem 3. Let $k \geq 2$ be a fixed positive integer, then for any positive integer group $(m_1,\ m_2,\ \cdots,\ m_k)$, we have the inequality

$$G(m_1 m_2 \cdots m_k) \leq G(m_1)G(m_2)\cdots G(m_k).$$

Note. For any positive integer n, we found that $n = 1,\ 4,\ 9$ are three positive integer solutions of the equation $G(n) = n$. Whether there exist infinite positive integers n such that the equation $G(n) = n$ is an interesting problem. We conjecture that the equation $G(n) = n$ has only three positive integer solutions $n = 1,\ 4,\ 9$. This is an unsolved problem.

§2. Proof of the theorem

In this section, we shall complete the proof of our theorems directely . First we prove Theorem 1. Let $G(p) = m$, namely $p \mid \prod_{k=1}^{m} \phi(k)$, $p \nmid \prod_{k=1}^{s} \phi(k), 0 < s < m$. If $m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$ be the factorization of m into prime powers, then $\phi(m) = q_1^{\alpha_1 - 1}(q_1 - 1)q_2^{\alpha_2 - 1}(q_2 - 1)\cdots q_r^{\alpha_r - 1}(q_r - 1)$, from the definition of $G(n)$ we can deduce that p divides one of $\phi(q_i^{\alpha_i})$, $1 \leq i \leq r$. So p divides $q_i^{\alpha_i - 1}$ or $q_i - 1$. We discuss it in the following two cases:

(i). If $p \mid q_i - 1$, then we must have $m = q_i = lp + 1$ be the smallest prime in the arithmetical series $\{kp + 1\}$.

(ii). If $p \mid q_i^{\alpha_i - 1}$, then we have have $\alpha_i = 2$, $m = q_i^2 = p^2$.

Combining (i) and (ii) we may immediately deduce that $G(p) = \min\{p^2,\ lp + 1\}$, where $lp + 1$ be the smallest prime in the arithmetical series $\{kp + 1\}$. Similarly, we can also deduce the calculating formulae for $G(p^2)$. This proves Theorem 1.

Now we prove Theorem 2. For any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers, $G(p_i^{\alpha_i}) = m_i$, $1 \leq i \leq r$, $m = \max\{m_1,\ m_2,\ \cdots,\ m_r\}$, then from the definition of $G(n)$ we have $p_i^{\alpha_i}$ divides $\phi(1)\phi(2)\cdots\phi(m_i)$ for all $1 \leq i \leq r$. So $p_i^{\alpha_i}$ divides $\phi(1)\phi(2)\cdots\phi(m)$. Since $(p_i,\ p_j) = 1$, $i \neq j$, so we must have $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ divides $\phi(1)\phi(2)\cdots\phi(m)$. Therefore, $G(n) = m = \max\{G(p_1^{\alpha_1}),\ G(p_2^{\alpha_2}),\ \cdots,\ G(p_r^{\alpha_r})\}$. So $G(n)$ is a Smarandache multiplicative function. From Theorem 1 we may immediately get

$$\sum_{n=1}^{\infty} \frac{G(n)}{n^2} > \sum_{p} \frac{G(p)}{p^2} > \sum_{p} \frac{1}{p} = \infty.$$

So the Dirichlet series $\sum_{n=1}^{\infty} \frac{G(n)}{n^2}$ is divergent.

Finally, we prove Theorem 3. First we prove that the inequality $G(m_1 m_2) \leq G(m_1)G(m_2)$ holds for any positive integer m_1 and m_2.

Let $G(m_1) = u$, $G(m_2) = v$, from the definition of $G(n)$ we can easily get

$$m_1 \mid \prod_{i=1}^{u} \phi(i)\ ,\ m_2 \mid \prod_{i=1}^{v} \phi(i).$$

Without loss of generality we can assume $u \leq v$, then

$$\prod_{k=1}^{uv}\phi(k) = \prod_{k=1}^{u}\phi(k)\cdot\prod_{k=u+1}^{uv}\phi(k) = \prod_{k=1}^{u}\phi(k)\cdot\phi(u+1)\cdots\phi(v)\cdots\phi(2v)\cdots\phi(uv)$$

Notice that

$$\phi(2)\mid\phi(2v),\ \phi(3)\mid\phi(3v),\ \cdots,\ \phi(u)\mid\phi(uv),\ \phi(u+1)\mid\phi(u+1),\ \cdots,\ \ \phi(v)\mid\phi(v),$$

this means that

$$\prod_{i=1}^{v}\phi(i)\mid\prod_{k=u+1}^{uv}\phi(k),$$

or

$$\prod_{i=1}^{u}\phi(i)\prod_{i=1}^{v}\phi(i)\mid\prod_{k=1}^{uv}\phi(k).$$

Hence

$$m_1m_2\mid\prod_{k=1}^{uv}\phi(k).$$

From the definition of $G(n)$ we know that $G(m_1m_2)\leq uv$, or

$$G(m_1m_2)\leq G(m_1)G(m_2).$$

If $k\geq 3$, then applying the above conclusion we can deduce that

$$\begin{aligned}
&G(m_1m_2\cdots m_k) = G(m_1(m_2\cdots m_k)) \leq G(m_1)G(m_2\cdots m_k)\\
&\leq G(m_1)G(m_2)G(m_3\cdots m_k)\\
&\cdots\cdots\cdots\\
&\leq G(m_1)G(m_2)\cdots G(m_k).
\end{aligned}$$

This completes the proof of Theorem 3.

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