ON THE SMARANDCHE FUNCTION AND ITS HYBRID MEAN VALUE

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Abstract For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined as the smallest $m \in N^+$ with $n|m!$. In this paper, we study the asymptotic property of a hybrid mean value of the Smarandache function and the Mangoldt function, and give an interesting hybrid mean value formula for it.

Keywords: the Smarandche function; the Mangoldt function; Mean value.

§1. Introduction

For any positive integer n, let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined as the smallest $m \in N^+$ with $n|m!$. From the definition of $S(n)$, one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime power factorization of n, then

$$S(n) = \max_{1 \le i \le k} S(p_i^{\alpha_i}).$$

About the arithmetical properties of $S(n)$, many people had studied it before (see reference [2]). In this paper, we study the asymptotic property of a hybrid mean value of the Smarandache function and the Mangoldt function, and give an interesting hybrid mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \ge 1$, we have the asymptotic formula

$$\sum_{n \le x} \wedge(n) S(n) = \frac{x^2}{4} + O\left(\frac{x^2 \log\log x}{\log x}\right),$$

where $\wedge(n)$ is the Mangoldt function defined by

$$\wedge(n) = \begin{cases} \log p, & \text{if } n = p^\alpha (\alpha \ge 1); \\ 0, & \text{otherwise.} \end{cases}$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Firstly, we need following:

Lemma. For any prime p and any positive integer α, we have

$$S(p^{\alpha}) = (p-1)\alpha + O\left(\frac{p\log\alpha}{\log p}\right).$$

Proof. From Theorem 1.4 of reference [3], we can obtain the estimate.

Now we use the above Lemma to complete the proof of the theorem. From the definition of $\wedge(n)$, we have

$$\begin{aligned}
&\sum_{n\le x}\wedge(n)S(n)\\
=&\sum_{p^{\alpha}\le x}S(p^{\alpha})\log p\\
=&\sum_{p\le x}\sum_{\alpha\le\frac{\log x}{\log p}}\log p\left((p-1)\alpha+O\left(\frac{p\log\alpha}{\log p}\right)\right)\\
=&\sum_{p\le x}(p-1)\log p\sum_{\alpha\le\frac{\log x}{\log p}}\alpha+O\left(\sum_{p\le x}p\sum_{\alpha\le\frac{\log x}{\log p}}\log\alpha\right).
\end{aligned}$$

Applying Euler's summation formula, we can get

$$\sum_{\alpha\le\frac{\log x}{\log p}}\alpha=\frac{1}{2}\frac{\log^2 x}{\log^2 p}+O\left(\frac{\log x}{\log p}\right),$$

and

$$\sum_{\alpha\le\frac{\log x}{\log p}}\log\alpha=\frac{\log x}{\log p}\log\frac{\log x}{\log p}-\frac{\log x}{\log p}+O\left(\log\frac{\log x}{\log p}\right).$$

Therefore we have

$$\sum_{n\le x}\wedge(n)S(n)=\frac{1}{2}\log^2 x\sum_{p\le x}\frac{p}{\log p}+O\left(\log x\log\log x\sum_{p\le x}\frac{p}{\log p}\right). \quad (1)$$

If $x>0$ let $\pi(x)$ denote the number of primes not exceeding x, and let

$$a(n)=\begin{cases}1, & \text{if } n \text{ is a prime;}\\ 0, & \text{otherwise.}\end{cases}$$

then $\pi(x)=\sum_{p\le x}a(n)$. Note the asymptotic formula

$$\pi(x)=\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right),$$

and from Abel's identity, we have

$$\begin{aligned}
&\sum_{p\le x}\frac{p}{\log p}\\
=\ &\sum_{n\le x}a(n)\frac{n}{\log n}\\
=\ &\pi(x)\frac{x}{\log x}-\pi(2)\frac{2}{\log 2}-\int_2^x\pi(t)d\left(\frac{t}{\log t}\right)\\
=\ &\frac{x}{\log x}\left(\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right)\right)-\int_2^x\left(\frac{t}{\log t}+O\left(\frac{t}{\log^2 t}\right)\right)d\left(\frac{t}{\log t}\right)\\
=\ &\frac{1}{2}\frac{x^2}{\log^2 x}+O\left(\frac{x^2}{\log^3 x}\right). \qquad (2)
\end{aligned}$$

Combining (1) and (2), we have

$$\begin{aligned}
&\sum_{n\le x}\wedge(n)S(n)\\
=\ &\frac{1}{4}x^2+O\left(\frac{x^2}{\log x}\right)+O\left(\log x\log\log x\frac{x^2}{\log^2 x}\right)\\
=\ &\frac{1}{4}x^2+O\left(\frac{x^2\log\log x}{\log x}\right).
\end{aligned}$$

This completes the proof of the theorem.

References

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