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On Fuzzy Matroids

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Abstract: The aim of this paper is to discuss properties of fuzzy regular-flats, fuzzy C-flats, fuzzy alternative-sets and fuzzy i-flats. Moreover, we characterize some peculiar fuzzy matroids via these notions. Finally, we provide a decomposition of fuzzy strong maps.

Key Words: Neutrosophic set, fuzzy matroid, fuzzy flat, fuzzy closure, fuzzy strong map, fuzzy hesitant map.

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§1. Introduction

The matroid theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of matroid problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The notion of fuzzy matroids was first introduced by Geotschel and Voxman in their landmark paper [4] using the notion of fuzzy independent set. The notion of fuzzy independent set was also explored in [10,9]. Some constructions, fuzzy spanning sets, fuzzy rank and fuzzy closure axioms were also studied in [5-7,13]. Several other fuzzifications of matroids were also discussed in [8,11]. Since the notion of flats in traditional matroids is one of the most significant notions that plays a very important rule in characterizing strong maps (see for example [3,12]). In [2], the notions of fuzzy flats and fuzzy closure flats were introduced and several examples were provided. Thus in [2], fuzzy matroids are defined via fuzzy flats axioms and it was shown that the levels of the fuzzy matroid introduced are indeed crisp matroids. Moreover, fuzzy strong maps and fuzzy hesitant maps are introduced and explored. We remark that the approach in [2] is different from those mentioned above. Let $FM = (E,\mathcal{O})$ be a fuzzy matroid. A fuzzy set $\lambda \in E$ is called a *fuzzy C-open set* in FM if there exists a fuzzy open set μ such that $\mu \leq \lambda \leq \bar{\mu}$ ([1]).

Let E be any non-empty set. A *neutrosophic set* based on neutrosophy, is defined for an element $x(T, I, F)$ belongs to the set if it is t true in the set, i indeterminate in the set, and f false, where t, i and f are real numbers taken from the sets T, I and F with no restriction on T, I, F nor on their sum $n = t + i + f$. Particularly, if $I = \emptyset$, we get the fuzzy set. By $\wp(1)$ we denote the set of all fuzzy sets on E. That is $\wp(1) = [0, 1]^E$, which is a completely distributive lattice. Thus let 0^E and 1^E denote its greatest and smallest elements, respectively. That is $0^E(e) = 0$ and $1^E(e) = 1$ for every $e \in E$. A fuzzy set μ_1 is a subset of μ_2 , written $\mu_1 \leq \mu_2$, if

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$\mu_1(e) \leq \mu_2(e)$ for all $e \in E$. If $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$, then μ_1 is a proper subset of μ_2, written $\mu_1 < \mu_2$. Moreover, $\mu_1 \prec \mu_2$ if $\mu_1 < \mu_2$ and there does not exist μ_3 such that $\mu_1 < \mu_3 < \mu_2$. Finally, $\mu_1 \vee \mu_2 = \sup\{\mu_1, \mu_2\}$ and $\mu_1 \wedge \mu_2 = \inf\{\mu_1, \mu_2\}$.

Next we recall some basic definitions and results from [2].

Definition 1.1 *Let E be a finite set and let $\mathfrak{F}$ be a family of fuzzy subsets of E satisfying the following three conditions:*

(*i*) $1^E \in \mathfrak{F}$;

(*ii*) *If $\mu_1, \mu_2 \in \mathfrak{F}$, then $\mu_1 \wedge \mu_2 \in \mathfrak{F}$;*

(*iii*) *If $\mu \in \mathfrak{F}$ and $\mu_1, \mu_2, ..., \mu_n$ are all minimal members of $\mathfrak{F}$ (with respect to standard fuzzy inclusion) that properly contain μ (in this case we write $\mu \prec \mu_i$ for all $i = 1, 2, ..., n$), then the fuzzy union of $\mu_1, \mu_2, ..., \mu_n$ is equal to 1^E (i.e. $\vee_{i=1}^{n} \mu_i = 1^E$). Then the system $FM = (E, \mathfrak{F})$ is called fuzzy matroid and the elements of $\mathfrak{F}$ are fuzzy flats of FM.*

Definition 1.2 *For $r \in (0, 1]$, let $C^r(\mu) = \{e \in E | \mu(e) \geq r\}$ be the r-level of a fuzzy set $\mu \in \mathfrak{F}$, and let $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ be the r-level of the family $\mathfrak{F}$ of fuzzy flats. Then for $r \in (0, 1]$, $(E, \mathfrak{F}^r)$ is the* r-level *of the fuzzy set system $(E, \mathfrak{F})$.*

Theorem 1.3 *For every $r \in (0, 1]$, $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ the r-levels of a family of fuzzy flats $\mathfrak{F}$ of a fuzzy matroid $FM = (E, \mathfrak{F})$ is a family of crisp flats.*

Definition 1.4 *Let E be any set with n-elements and $\mathfrak{F} = \{\chi_A : A \leq E,\ |A| = n \text{ or } |A| < m\}$ where m is a positive integer such that $m \leq n$. Then $(E, \mathfrak{F})$ is a fuzzy matroid called the fuzzy uniform matroid on n-elements and rank m, denoted by $F_{m,n}$. $F_{m,m}$ is called the free fuzzy uniform matroid on n-elements.*

We remark that the rank notion in the preceding definition coincides with that in [6].

Definition 1.5 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and $\mu \in \mathfrak{F}$. Then the fuzzy closure of μ is $\bar{\mu} = \bigwedge_{\lambda \in \mathfrak{F}, \mu \leq \lambda} \lambda$.*

Theorem 1.6 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and X be a non-empty subset of E. Then $(X, \mathfrak{F}_X)$ is a fuzzy matroid, where $\mathfrak{F}_X = \{\chi_X \wedge \mu : \mu \in \mathfrak{F}\}$.*

Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid, X be a non-empty subset of E and μ be a fuzzy set in X. We may realize μ as a fuzzy set in E by the convention that $\mu(e) = 0$ for all $e \in E - X$. It can be easily shown that $\mathfrak{F}_X = \{\mu|_X : \mu \in \mathfrak{F}\}$, where $\mu|_X$ is the restriction of μ to X.

Let E_1 and E_2 be two sets, μ_1 is a fuzzy set in E_1, μ_2 is a fuzzy set in E_2 and $f : E_1 \to E_2$ be a map. Then we define the fuzzy sets $f(\mu_1)$ (the image of μ_1) and $f^{-1}(\mu_2)$ (the preimage of μ_2) by

$$f(\mu_1)(y) = \begin{cases} \sup\{\mu_1(x) : x \in f^{-1}(\{y\})\} & , y \in \text{Range}(f) \\ 1 & , \text{Otherwise}, \end{cases}$$

and $f^{-1}(\mu_2)(x) = \mu_2(f(x))$ for all $x \in E_1$.

Definition 1.7 *A fuzzy strong map from a fuzzy matroid $FM_1 = (E_1, \mathfrak{F}_1)$ into a fuzzy matroid $FM_2 = (E_2, \mathfrak{F}_2)$ is a map $f : E_1 \to E_2$ such that the preimage of every fuzzy flat in FM_2 is a fuzzy flat in FM_1.*

Theorem 1.8 *Let $FM_1 = (E_1, \mathfrak{F}_1)$ and $FM_2 = (E_2, \mathfrak{F}_2)$ be fuzzy matroids and $f : E_1 \to E_2$ be a map. Then the following are equivalent:*

(*i*) *f is fuzzy strong;*
(*ii*) *For every fuzzy set μ_1 in FM_1, $f(\overline{\mu_1}) \le \overline{f(\mu_1)}$;*
(*iii*) *For every fuzzy set μ_2 in FM_2, $\overline{f^{-1}(\mu_2)} \le f^{-1}(\overline{\mu_2})$.*

Next, we recall some results from [1].

Definition 1.9 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. Then μ is a fuzzy c-flat if $\bigvee_{\lambda \in \mathfrak{F}, \lambda \le \overline{\mu}} \lambda \le \mu$.*

Clearly, every fuzzy flat is a fuzzy c-flat, but the converse need not be true.

Example 1.1 Let $E = \{a, b, c, d\}$ and $\mathfrak{F} = \{1^E, 0, \chi_{\{a,b\}}, \chi_{\{c,d\}}\}$. $FM = (E, \mathfrak{F})$ is a fuzzy matroid. $\chi_{\{b,d\}}$ is a fuzzy c- flat that is not a fuzzy flat.

Lemma 1.10 *The intersection of fuzzy c-flats is a fuzzy c-flat.*

Lemma 1.11 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. The fuzzy C closure of μ is $\overline{\mu}^F = \bigwedge\{\acute{\mu} : \acute{\mu}$ is a fuzzy c-flat and $\mu \le \acute{\mu}\}$.*

Theorem 1.12 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ, λ be fuzzy sets. Then*

i) $\overline{0}^F = 0$;
ii) *$\overline{\mu}^F$ is a fuzzy closure flat;*
iii) $\mu \le \overline{\mu}^F$;
iv) *If $\mu \le \lambda$, then $\overline{\mu}^F \le \overline{\lambda}^F$;*
v) $\overline{\overline{\mu}^F}^F = \overline{\mu}^F$.

Lemma 1.13 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. Then μ is a fuzzy c-flat if and only if $\overline{\mu}^F = \mu$.*

Lemma 1.14 *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ, λ be fuzzy sets. Then*

i) $\overline{\mu \vee \lambda}^F \ge \overline{\mu}^F \vee \overline{\lambda}^F$;
ii) $\overline{\mu \wedge \lambda}^F \le \overline{\mu}^F \wedge \overline{\lambda}^F$.

Definition 1.15 *A map $f : FM_1 \to FM_2$ is*

i) *fuzzy c-strong if the inverse image of every fuzzy flat of FM_2 is a fuzzy c-flat of FM_1;*
ii) *fuzzy hesitant if the inverse image of every fuzzy c-flat of FM_2 is a fuzzy c-flat of FM_1.*

Clearly, a fuzzy strong (fuzzy hesitant) map is fuzzy c-strong, but the converse need not be true since a fuzzy c-flat need not be a fuzzy flat as we have seen in Example 1.1.

A map $f : FM_1 \to FM_2$ is said to be *fuzzy* if the image of every fuzzy flat of FM_1 is a fuzzy flat of FM_2. The following is a trivial result.

Lemma 1.16 *Let $f : FM_1 \to FM_2$ be a fuzzy map that is also fuzzy strong. Then $f^{-1}(\bar{\mu}) = \overline{f^{-1}(\mu)}$ for every fuzzy set μ of FM_2.*

Theorem 1.17 *A fuzzy map $f : FM_1 \to FM_2$ that is also fuzzy strong is fuzzy hesitant.*

Theorem 1.18 *The following are equivalent for a map $f : FM_1 \to FM_2$:*

i) *f is hesitant;*
ii) $f(\overline{\mu}^F) \leq \overline{f(\mu)}^F$ *for every fuzzy set μ of FM_1;*
iii) $\overline{f^{-1}(\lambda)}^F \leq f^{-1}(\overline{\lambda}^F)$ *for every fuzzy set λ of FM_2.*

§2. Fuzzy-Regular- and Fuzzy i-Flats

In this section, the notions of fuzzy–regular-flat and fuzzy-i-flats are discussed. We prove that the notion of fuzzy-i-flat coincides with that of fuzzy-c-flat. In addition, we provide several characterizations of fuzzy-regular-flats and fuzzy open sets of certain fuzzy matroids.

Definition 2.1 *Let $FM = (E,\mathcal{O})$ be a fuzzy matroid. A fuzzy set λ is* nowhere-spanning boundary *if* $\overset{o}{\overline{\bar{\lambda}\backslash\overset{o}{\lambda}}} = 0$, fuzzy local-flat *if* $\lambda = \mu \wedge \bar{\lambda}$, *where μ is fuzzy open and λ is fuzzy C-preopen if* $\mu \leq \overset{o}{\bar{\mu}}$ *and* fuzzy-i-flat *if* $\overset{o}{\bar{\lambda}} = \overset{o}{\lambda}$.

The following example shows that a nowhere-spanning-boundary-fuzzy set needs not be a fuzzy-c-flat. In the next coming theorem we prove a partial converse of this.

Example 2.1 Let $E = \{a, b, c\}$ and $\mathcal{O} = \{1, \chi_{\{b,c\}}\}$. Then $\chi_{\{a,b\}}$ is an nowhere-spanning-boundary-fuzzy set but not a fuzzy-c-flat.

Theorem 2.2 *In a loopless fuzzy matroid, every fuzzy-c-flat is a nowhere-spanning-boundary-fuzzy set.*

Proof Clearly the intersection of two nowhere-spanning-boundary-fuzzy sets is a nowhere-spanning-boundary-fuzzy set. Since a fuzzy-c-flat is an intersection of a (fuzzy C-open) fuzzy open set and a fuzzy closure flat, it is enough to show that every fuzzy C-open and every fuzzy c-flat is a nowhere-spanning-boundary-fuzzy set. If λ is a C-open, then for some fuzzy open set μ we have $\mu \leq \lambda \leq \bar{\mu}$. Since $\bar{\lambda}\backslash\overset{o}{\lambda} \leq \bar{\mu}\backslash\overset{o}{\mu} = 0$. Thus $\overset{o}{\overline{\bar{\lambda}\backslash\overset{o}{\lambda}}} = 0$. In fact, it is obvious that every fuzzy open set is a nowhere-spanning-boundary-fuzzy set. Thus fuzzy C–open- (and hence every fuzzy C–flat-) set is a nowhere-spanning-boundary-fuzzy set. □

The following result shows that the fuzzy-i-flats class coincides with the class of closure flats.

Theorem 2.3 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy matroid. Then the following are equivalent:*

(1) *λ is a fuzzy-i-flat;*

(2) *λ is a closure flat;*

(3) *$1 - \lambda$ is a C-preopen and λ is a fuzzy-c-flat;*

(4) *$1 - \lambda$ is a C-preopen and λ is a nowhere-spanning-boundary-set.*

Proof (1) $\Rightarrow$ (2) Since $\overset{o}{\bar{\lambda}} = \overset{o}{\lambda} \leq \lambda$, then $1 - \lambda \leq \overline{(1-\lambda)^o}$. Thus $1 - \lambda$ is fuzzy C-open, hence isλ a fuzzy-c-flat.

(2) $\Rightarrow$ (3) Every fuzzy-c-flat is trivially fuzzy C-preopen. Since $\lambda = 1 \wedge \lambda$, where 1 is fuzzy open and λ is a fuzzy-c-flat, then $1 - \lambda$ is an fuzzy-c-flat.

(3) $\Rightarrow$ (4) Theorem 2.2.

(4) $\Rightarrow$ (1) Since λ is a nowhere-spanning-boundary-fuzzy set, $\mu = 1 - \lambda$ is also a nowhere-spanning-boundary-fuzzy set and as

$$(\bar{\mu} \backslash \backslash \overset{o}{\mu})^o = \overset{o}{\bar{\mu}} \wedge \overset{o}{\overline{1-\mu}} = \overset{o}{\bar{\mu}} \wedge (1 - \overline{\overset{o}{\mu}}) = \overset{o}{\bar{\mu}} \backslash \overline{\overset{o}{\mu}},$$

it follows that $\overset{o}{\bar{\mu}} \leq \overline{\overset{o}{\mu}}$. Since μ is fuzzy C-preopen, $\mu \leq \overline{\overset{o}{\bar{\mu}}}$. Thus $\mu \leq \overline{\overset{o}{\mu}}$ or equivalently $\bar{\mu} = \overline{\overset{o}{\mu}}$. Since $\mu = 1 - \lambda$, $\overset{o}{\bar{\lambda}} = \overset{o}{\lambda}$. □

A matroid $M = (E, \mathcal{O})$ is called *a fuzzy closure matroid* if $\overline{\lambda \vee \mu} = \bar{\lambda} \vee \bar{\mu}$ for all fuzzy subsets λ and μ of E. Next we characterize the class of fuzzy-regular-flats of a fuzzy closure matroid. We show that the class of fuzzy-regular-flats is the intersection of the class of fuzzy local-flats with either the class of fuzzy C-open-sets or the class of fuzzy C-preopen-sets.

Theorem 2.4 *Let $M = (E, \mathcal{O})$ be a fuzzy closure matroid and $\lambda \in E$. Then the following are equivalent:*

(1) *λ is a fuzzy-regular-flat;*

(2) *λ is fuzzy C–open set and a fuzzy local-flat;*

(3) *λ is fuzzy C-preopen-set and a fuzzy local-flat.*

Proof (1) $\Rightarrow$ (2) Every fuzzy-regular-flat is clearly a fuzzy local-flat. Let $\lambda = \mu \wedge \eta$ be an fuzzy-regular-flat, where μ is fuzzy open and η is a fuzzy flat such that $\eta = \overline{\overset{o}{\eta}}$. Since $\lambda = \mu \wedge \eta$, we have $\mu \wedge \eta^o \leq \overset{o}{\lambda}$. It is easily seen that $\overset{o}{\lambda} \leq \lambda \leq \eta$, hence $\overset{o}{\lambda} \leq \eta^o$. But $\overset{o}{\lambda} \leq \lambda \leq \mu$, hence $\overset{o}{\lambda} \leq \mu \wedge \eta^o$. Therefore $\overset{o}{\lambda} = \mu \wedge \eta^o$. Now we prove $\lambda \leq \overline{\lambda^o}$. Let $e \in \lambda$ and δ be an fuzzy open set containing e. Then $e \in \mu \wedge \delta = (1 - \lambda_1) \wedge (1 - \lambda_2)$ for some fuzzy flats λ_1 and λ_2. Thus $e \in 1 - (\lambda_1 \vee \lambda_2) = 1 - (\overline{\lambda_1} \vee \overline{\lambda_2}) = 1 - (\overline{\lambda_1 \vee \lambda_2})$ which is fuzzy open. Since $e \in \eta = \overline{\overset{o}{\eta}}$, there exists $l \in \overset{o}{\eta}$ such that $l \neq e$ and $l \in 1 - (\overline{\lambda_1 \vee \lambda_2}) = \mu \wedge \eta$. Hence $l \in \mu \wedge \eta^o = \overset{o}{\lambda}$. Therefore $e \in \overline{\overset{o}{\lambda}}$ and $\lambda \leq \overline{\overset{o}{\lambda}}$. From $\overset{o}{\lambda} \leq \lambda \leq \overline{\overset{o}{\lambda}}$ we know that λ is fuzzy C-open.

(2) $\Rightarrow$ (3) is trivial

(3) $\Rightarrow$ (1) Since λ is a fuzzy local-flat, $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. As λ is fuzzy C-preopen and as $\overline{\overset{o}{\lambda}} \leq \bar{\lambda}$, $\bar{\lambda}$ is a fuzzy regular-flat. Thus λ is an fuzzy-regular-flat. □

Next, we characterize the class of fuzzy open sets of a loopless closure fuzzy matroid, hence we characterize the class of fuzzy flats.

Theorem 2.5 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid and $\lambda \in E$. Then the following are equivalent:*

(1) *λ is fuzzy open;*
(2) *λ is fuzzy prespanning and a fuzzy local-flat;*
(3) *λ is fuzzy prespanning and an fuzzy-regular-flat;*
(4) *λ is fuzzy prespanning and an fuzzy-c-flat.*

Proof (1) $\Rightarrow$ (2) Since $\lambda \leq \bar{\lambda}$, $\lambda = \overset{o}{\lambda} \leq \overset{o}{\bar{\lambda}}$. Thus λ is fuzzy prespanning. As 1 is a fuzzy flat and $\lambda = \overset{o}{\lambda} \wedge 1$, λ is a fuzzy local-flat.

(2) $\Rightarrow$ (3) Since λ is a fuzzy local-flat, $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. Since $\overline{\overset{o}{\bar{\lambda}}} \leq \overline{\lambda}$, $\overline{\overset{o}{\bar{\lambda}}} \leq \overline{\lambda}$. But as λ is fuzzy prespanning, $\lambda \leq \overset{o}{\bar{\lambda}}$ and thus $\overline{\lambda} \leq \overline{\overset{o}{\bar{\lambda}}}$. Hence $\overline{\lambda}$ is a fuzzy regular-flat and so λ is a fuzzy-regular-flat.

(3) $\Rightarrow$ (4) Clearly a fuzzy flat is a fuzzy-i-flat and thus a fuzzy-regular-flat is an fuzzy-c-flat.

(4) $\Rightarrow$ (1) Since λ is an fuzzy-c-flat, we have $\lambda = \mu \wedge \eta$ where μ is open and $\overset{o}{\bar{\eta}} = \overset{o}{\eta}$. Because λ is fuzzy prespanning, we have $\lambda \leq \overset{o}{\bar{\lambda}} = \overset{o}{\overline{(\mu \wedge \eta)}} \subseteq \overset{o}{\bar{\mu}} \wedge \overset{o}{\bar{\eta}}$. Hence $\lambda = (\mu \wedge \eta) \wedge \mu \leq \mu \wedge \overset{o}{\eta}$. Notice $\lambda = \mu \wedge \eta \geq \mu \wedge \overset{o}{\eta}$, we have $\lambda = \mu \wedge \overset{o}{\eta}$. Thus as M is a closure fuzzy matroid, λ is fuzzy open.$\Box$

§3. Characterizations of Particular Fuzzy Matroids

In this section, we characterize maximal fuzzy matroids, local-flat-fuzzy matroids, free fuzzy matroids and others via fuzzy-regular-flats and fuzzy-c-flats. We provide a decomposition of fuzzy strong maps at the end of this section.

Theorem 3.1 *For a loopless fuzzy matroid $M = (E, \mathcal{O})$, the following are equivalent:*

(1) *M is maximal;*
(2) *Every fuzzy subset of E is an fuzzy-c-flat;*
(3) *Every spanning fuzzy subset of E is an fuzzy-c-flat.*

Proof (1) $\Rightarrow$ (2) Let $\lambda \in E$. Since every submatroid of a maximal fuzzy matroid is maximal, then $M|\bar{\lambda}$ is maximal. Since $\underline{\lambda}$ is a spanning fuzzy set in $M|\bar{\lambda}$, λ is fuzzy open in $M|\bar{\lambda}$. Thus $\lambda = \mu \wedge \bar{\lambda}$ where μ is a fuzzy open set in M and $\bar{\lambda}$ is a fuzzy-c-flat in M. Hence λ is an fuzzy-c-flat.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let $\bar{\lambda} = 1$. By (3) $\lambda = \mu \wedge \eta$, where μ is fuzzy open and η is a fuzzy-c-flat. Since $\lambda \leq \eta$, $\bar{\eta} = 1$ and hence $\overset{o}{\eta} = \overset{o}{\bar{\eta}} = \overset{o}{1} = 1$, since M is loopless. Thus $\eta = 1$ and $\lambda = \mu$ is fuzzy open. Therefore, M is maximal. $\Box$

Theorem 3.2 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid. Then the following are* *equivalent:*

(1) *M is a locally-couniform fuzzy matroid;*
(2) *Every fuzzy-c-flat is both fuzzy open and a fuzzy flat;*
(3) *Every fuzzy-c-flat is a fuzzy flat.*

Proof (1) $\Rightarrow$ (2) If λ is an fuzzy-c-flat, $\lambda = \mu \wedge \eta$, where μ is fuzzy open and η is a C-fuzzy flat. By (1) μ is also a fuzzy flat. On the other hand η is fuzzy open by (1) and thus $\overset{o}{\overline{\eta}} \leq \eta \leq \eta$ and so η is both fuzzy open and a fuzzy flat. Therefore, λ is a fuzzy flat being the intersection of two fuzzy flats and as M is a closure fuzzy matroid, λ is also fuzzy open.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Every fuzzy open set is a fuzzy-c-flat by Theorem 2.5 and thus by (3) a fuzzy flat. □

Theorem 3.3 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid. Then the following are equivalent:*

(1) *$M \cong F_{1,n}$ for some positive integer $n \geq 1$;*
(2) *The only fuzzy-c-flats in M are the trivial ones;*
(3) *The only fuzzy-regular-flats in M are the trivial ones.*

Proof (1) $\Rightarrow$ (2) If λ is an fuzzy-c-flat, then $\lambda = \mu \wedge \eta$, where μ is fuzzy open and η is a fuzzy-c-flat ($\eta^o = \overline{\eta^o}$). If $\lambda \neq 0$, then $\mu \neq 0$ and by (1) $\mu = 1$. Thus $\lambda = \eta$ and so $\lambda^o = \overset{o}{\bar{\lambda}} = 1^o = 1$. Hence $\lambda = 1$.

(2) $\Rightarrow$ (3) Every fuzzy-regular-flat is an fuzzy-c-flat.

(3) $\Rightarrow$ (1) Since every fuzzy open set is a fuzzy-regular-flat, by (3) the only fuzzy open sets in M are the trivial ones. □

It is well-known that the notions of fuzzy-regular-flat and fuzzy C-open-set are independent from each other. By Theorem 2.4 in a fuzzy closure matroid, every fuzzy-regular-flat is fuzzy C-open. Clearly a fuzzy-regular-flat is a fuzzy local-flat in any fuzzy matroid. Next we show that a fuzzy C-open-set which is also a fuzzy local-flat has to be a fuzzy-regular-flat.

Theorem 3.4 *In any fuzzy matroid, every fuzzy set λ that is both fuzzy C-open and a fuzzy local-flat is a fuzzy-regular-flat.*

Proof Since λ is fuzzy C-open, $\lambda \leq \overset{\overline{o}}{\lambda}$ and since λ is a fuzzy local-flat, $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. Then $\bar{\lambda} = \overset{\overline{o}}{\lambda}$ and so $\overline{\lambda}$ is a fuzzy regular-flat. Hence λ is a fuzzy-regular-flat. □

Corollary 3.5 *Let $M = (E, \mu)$ be a fuzzy closure matroid and $\lambda \leq 1$. Then λ is a fuzzy-regular-flat if and only if λ is both fuzzy C-open-set and a fuzzy local-flat.*

Theorem 3.6 *Let $M = (E, \mu)$ be a loopless fuzzy closure matroid. Then M is free if and only if every fuzzy subset of E is a fuzzy-regular-flat.*

Proof Let M be free. Then every fuzzy set $\lambda \leq 1$ is open and a fuzzy regular-flat. Hence

λ is a fuzzy-regular-flat.

Conversely if every fuzzy subset of E is a fuzzy-regular-flat, then every singleton $e \leq 1$ is a fuzzy-regular-flat and by Theorem 2.4 fuzzy C-open. If $e^o = 0$, then we have the contradiction $e \leq \overline{e^o} = 0$. Thus $e = e^o$ or equivalently every singleton is fuzzy open. Thus every fuzzy subset of E is fuzzy open and hence M is free. □

Theorem 3.7 *Let $M = (E, \mathcal{O})$ be a loopless fuzzy closure matroid and $\lambda \leq 1$. Then the following are equivalent:*

(1) *λ is fuzzy open;*

(2) *λ is a alternative-fuzzy set and a fuzzy local-flat;*

(3) *λ is fuzzy prespanning and a fuzzy local-flat.*

Proof (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (1) Let λ be a fuzzy prespanning set that is also a fuzzy local-flat. Then $\lambda \leq \overset{o}{\bar{\lambda}}$ and $\lambda = \mu \wedge \bar{\lambda}$, where μ is fuzzy open. Thus $\lambda \leq \mu \wedge \overset{o}{\bar{\lambda}} = (\mu \wedge \bar{A})^o = \overset{o}{\lambda}$. Therefore, λ is fuzzy open. □

Definition 3.8 *A fuzzy map $f : M_1 \to M_2$ is called* fuzzy hesitant *(resp.* fuzzy alternative-strong, fuzzy prestrong, fuzzy local–flat-strong, fuzzy open–regular-flat-strong*) if the inverse image of every open set in M_2 is a fuzzy C-open (resp. fuzzy alternative-set, fuzzy prespanning set, fuzzy local-flat, fuzzy-regular-flat) in M_1.*

Combining Corollary 3.5 and Theorem 3.4, we immediately obtain the following decomposition of fuzzy strong maps.

Theorem 3.9 *Let $f : M_1 \to M_2$ be a fuzzy map where M_1 is a loopless fuzzy closure matroid. Then*

(*i*) *f is fuzzy open–regular-flat-strong if and only if f is fuzzy hesitant and fuzzy local-flat-strong;*

(*ii*) *f is fuzzy strong if and only if f is fuzzy alternative-strong and fuzzy local-flat-strong;*

(*iii*) *f is fuzzy alternative-strong if and only if f is fuzzy prestrong and fuzzy hesitant;*

(*iv*) *f is fuzzy strong if and only if f is fuzzy prestrong and fuzzy local-flat-strong;*

(*v*) *f is fuzzy strong if and only if f is fuzzy prestrong and fuzzy open–regular-flat-strong.*

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