International J.Math. Combin. Vol.1 (2010), 115-121

Euler-Savary's Formula for the Planar Curves in Two Dimensional Lightlike Cone

Handan BALGETİR ÖZTEKİN and Mahmut ERGÜT

(Department of Mathematics, Fırat University, 23119 Elazığ, TÜRKİYE)

Email: handanoztekin@gmail.com & mergut@firat.edu.tr

Abstract: In this paper, we study the Euler-Savary's formula for the planar curves in the lightlike cone. We first define the associated curve of a curve in the two dimensional lightlike cone Q^2.Then we give the relation between the curvatures of a base curve, a rolling curve and a roulette which lie on two dimensional lightlike cone Q^2.

Keywords: Lightlike cone, Euler Savary's formula, Smarandache geometry, Smarandachely denied-free.

AMS(2010): 53A04, 53A30, 53B30

§1. Introduction

The Euler-Savary's Theorem is well known theorem which is used in serious fields of study in engineering and mathematics.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways. So the Euclidean geometry is just a Smarandachely denied-free geometry.

In the Euclidean plane E^2,let c_B and c_R be two curves and P be a point relative to c_R. When c_R roles without splitting along c_B, the locus of the point P makes a curve $c_{L.}$ The curves c_B, c_R and c_L are called the base curve, rolling curve and roulette, respectively. For instance, if c_B is a straight line, c_R is a quadratic curve and P is a focus of c_R, then c_L is the Delaunay curve that are used to study surfaces of revolution with the constant mean curvature, (see [1]).The relation between the curvatures of this curves is called as the Euler-Savary's formula.

Many studies on Euler-Savary's formula have been done by many mathematicians. For example, in [4], the author gave Euler-Savary's formula in Minkowski plane. In [5], they expressed the Euler-Savary's formula for the trajectory curves of the 1-parameter Lorentzian spherical motions.

On the other hand, there exists spacelike curves, timelike curves and lightlike(null) curves in semi-Riemannian manifolds. Geometry of null curves and its applications to general reletivity in semi-Riemannian manifolds has been constructed, (see [2]). The set of all lightlike(null)

[1]Received February 1, 2010. Accepted March 30,2010.

vectors in semi-Riemannian manifold is called the lightlike cone. We know that it is important to study submanifolds of the lightlike cone because of the relations between the conformal transformation group and the Lorentzian group of the n-dimensional Minkowski space E_1^n and the submanifolds of the n-dimensional Riemannian sphere S^n and the submanifolds of the (n+1)-dimensional lightlike cone Q^{n+1}. For the studies on lightlike cone, we refer [3].

In this paper, we have done a study on Euler-Savary's formula for the planar curves in two dimensional lightlike cone Q^2. However, to the best of author's knowledge, Euler-Savary's formula has not been presented in two dimensional lightlike cone Q^2. Thus, the study is proposed to serve such a need. Thus, we get a short contribution about Smarandache geometries.

This paper is organized as follows. In Section2, the curves in the lightlike cone are reviewed. In Section3, we define the associated curve that is the key concept to study the roulette, since the roulette is one of associated curves of the base curve. Finally, we give the Euler-Savary's formula in two dimensional cone Q^2.

We hope that, these study will contribute to the study of space kinematics, mathematical physics and physical applications.

§2. Euler-Savary's Formula in the Lightlike Cone $\mathbf{Q}^2$

Let E_1^3 be the 3−dimensional Lorentzian space with the metric

$$g(x, y) = \langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in E_1^3$.

The lightlike cone Q^2 is defined by

$$Q^2 = \{x \in E_1^3 : g(x, x) = 0\}.$$

Let $x : I \to Q^2 \subset E_1^3$ be a curve, we have the following Frenet formulas (see [3])

$$\begin{aligned} x'(s) &= \alpha(s) \\ \alpha'(s) &= \kappa(s)x(s) - y(s) \\ y'(s) &= -\kappa(s)\alpha(s), \end{aligned} \tag{2.1}$$

where s is an arclength parameter of the curve $x(s)$. $\kappa(s)$ is cone curvature function of the curve $x(s)$,and $x(s)$, $y(s)$, $\alpha(s)$ satisfy

$$\begin{aligned} \langle x, x\rangle &= \langle y, y\rangle = \langle x, \alpha\rangle = \langle y, \alpha\rangle = 0, \\ \langle x, y\rangle &= \langle \alpha, \alpha\rangle = 1. \end{aligned}$$

For an arbitrary parameter t of the curve $x(t)$, the cone curvature function κ is given by

$$\kappa(t) = \frac{\left\langle \frac{dx}{dt}, \frac{d^2x}{dt^2}\right\rangle^2 - \left\langle \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2}\right\rangle \left\langle \frac{dx}{dt}, \frac{dx}{dt}\right\rangle}{2\left\langle \frac{dx}{dt}, \frac{dx}{dt}\right\rangle^5} \tag{2.2}$$

Using an orthonormal frame on the curve $x(s)$ and denoting by $\overline{\kappa}$, $\overline{\tau}$, β and γ the curvature, the torsion, the principal normal and the binormal of the curve $x(s)$ in E_1^3, respectively, we have

$$\begin{aligned} x' &= \alpha \\ \alpha' &= \kappa x - y = \overline{\kappa}\beta, \end{aligned}$$

where $\kappa \neq 0$, $\langle \beta, \beta \rangle = \varepsilon = \pm 1$, $\langle \alpha, \beta \rangle = 0, \langle \alpha, \alpha \rangle = 1$, $\varepsilon\kappa < 0$. Then we get

$$\beta = \varepsilon \frac{\kappa x - y}{\sqrt{-2\varepsilon\kappa}}, \quad \varepsilon\overline{\tau}\gamma = \frac{\kappa'}{2\sqrt{-2\varepsilon\kappa}}(x + \frac{1}{\kappa}y). \tag{2.3}$$

Choosing

$$\gamma = \sqrt{\frac{-\varepsilon\kappa}{2}}(x + \frac{1}{\kappa}y), \tag{2.4}$$

we obtain

$$\overline{\kappa} = \sqrt{-2\varepsilon\kappa}, \quad \overline{\tau} = -\frac{1}{2}(\frac{\kappa'}{\kappa}). \tag{2.5}$$

Theorem 2.1 *The curve $x : I \to Q^2$ is a planar curve if and only if the cone curvature function κ of the curve $x(s)$ is constant [3].*

If the curve $x : I \to Q^2 \subset E_1^3$ is a planar curve, then we have following Frenet formulas

$$\begin{aligned} x' &= \alpha, \\ \alpha' &= \varepsilon\sqrt{-2\varepsilon\kappa}\beta, \\ \beta' &= -\sqrt{-2\varepsilon\kappa}\alpha. \end{aligned} \tag{2.6}$$

Definition 2.2 *Let $x : I \to Q^2 \subset E_1^3$ be a curve with constant cone curvature κ (which means that x is a conic section) and arclength parameter s. Then the curve*

$$x_A = x(s) + u_1(s)\alpha + u_2(s)\beta \tag{2.7}$$

is called the associated curve of $x(s)$ in the Q^2, where $\{\alpha, \beta\}$ is the Frenet frame of the curve $x(s)$ and $\{u_1(s), u_2(s)\}$ is a relative coordinate of $x_A(s)$ with respect to $\{x(s), \alpha, \beta\}$.

Now we put

$$\frac{dx_A}{ds} = \frac{\delta u_1}{ds}\alpha + \frac{\delta u_2}{ds}\beta. \tag{2.8}$$

Using the equation (2.2) and (2.6), we get

$$\frac{dx_A}{ds} = (1 + \frac{du_1}{ds} - \sqrt{-2\varepsilon\kappa}u_2)\alpha + (u_1\varepsilon\sqrt{-2\varepsilon\kappa} + \frac{du_2}{ds})\beta. \tag{2.9}$$

Considering the (2.8) and (2.9), we have

$$\begin{aligned} \frac{\delta u_1}{ds} &= (1 + \frac{du_1}{ds} - \sqrt{-2\varepsilon\kappa}u_2) \\ \frac{\delta u_2}{ds} &= (u_1\varepsilon\sqrt{-2\varepsilon\kappa} + \frac{du_2}{ds}) \end{aligned} \tag{2.10}$$

Let s_A be the arclength parameter of x_A. Then we write

$$\frac{dx_A}{ds} = \frac{dx_A}{ds_A}.\frac{ds_A}{ds} = v_1\alpha + v_2\beta \tag{2.11}$$

and using (2.8) and (2.10), we get

$$\begin{aligned} v_1 &= 1 + \frac{du_1}{ds} - \sqrt{-2\varepsilon\kappa}u_2 \\ v_2 &= u_1\varepsilon\sqrt{-2\varepsilon\kappa} + \frac{du_2}{ds}. \end{aligned} \tag{2.12}$$

The Frenet formulas of the curve x_A can be written as follows:

$$\begin{aligned} \frac{d\alpha_A}{ds_A} &= \varepsilon_A\sqrt{-2\varepsilon_A\kappa_A}\beta_A \\ \frac{d\beta_A}{ds_A} &= -\sqrt{-2\varepsilon_A\kappa_A}\alpha_A, \end{aligned} \tag{2.13}$$

where κ_A is the cone curvature function of x_A and $\varepsilon_A = \langle\beta_A, \beta_A\rangle = \pm 1$ and $\langle\alpha_A, \alpha_A\rangle = 1$.

Let θ and ω be the slope angles of x and x_A respectively. Then

$$\overline{\kappa}_A = \frac{d\omega}{ds_A} = (\overline{\kappa} + \frac{d\phi}{ds})\frac{1}{\sqrt{|v_1^2 + \varepsilon v_2^2|}}, \tag{2.14}$$

where $\phi = \omega - \theta$.

If β is spacelike vector, then we can write

$$\cos\phi = \frac{v_1}{\sqrt{v_1^2 + v_2^2}} \quad \text{and} \quad \sin\phi = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}.$$

Thus, we get

$$\frac{d\phi}{ds} = \frac{d}{ds}(\cos^{-1}\frac{v_1}{\sqrt{v_1^2 + v_2^2}})$$

and (2.14) reduces to

$$\overline{\kappa}_A = (\overline{\kappa} + \frac{v_1 v_2' - v_1' v_2}{v_1^2 + v_2^2})\frac{1}{\sqrt{v_1^2 + v_2^2}}.$$

If β is timelike vector, then we can write

$$\cosh\phi = \frac{v_1}{\sqrt{v_1^2 - v_2^2}} \quad \text{and} \quad \sinh\phi = \frac{v_2}{\sqrt{v_1^2 - v_2^2}}$$

and we get

$$\frac{d\phi}{ds} = \frac{d}{ds}(\cosh^{-1}\frac{v_1}{\sqrt{v_1^2 - v_2^2}}).$$

Thus, we have

$$\overline{\kappa}_A = (\overline{\kappa} + \frac{v_1 v_2' - v_1' v_2}{v_1^2 - v_2^2})\frac{1}{\sqrt{v_1^2 - v_2^2}}.$$

Let x_B and x_R be the base curve and rolling curve with constant cone curvature κ_B and κ_R in Q^2, respectively. Let P be a point relative to x_R and x_L be the roulette of the locus of P.

We can consider that x_L is an associated curve of x_B such that x_L is a planar curve in Q^2, then the relative coordinate $\{w_1, w_2\}$ of x_L with respect to x_B satisfies

$$\begin{aligned}\frac{\delta w_1}{ds_B} &= 1+\frac{dw_1}{ds_B}-\sqrt{-2\varepsilon_B\kappa_B}w_2\\ \frac{\delta w_2}{ds_B} &= w_1\varepsilon_B\sqrt{-2\varepsilon_B\kappa_B}+\frac{dw_2}{ds_B}\end{aligned}\tag{2.15}$$

by virtue of (2.10).

Since x_R roles without splitting along x_B at each point of contact, we can consider that $\{w_1, w_2\}$ is a relative coordinate of x_L with respect to x_R for a suitable parameter s_R. In this case, the associated curve is reduced to a point P. Hence it follows that

$$\begin{aligned}\frac{\delta w_1}{ds_R} &= 1+\frac{dw_1}{ds_R}-\sqrt{-2\varepsilon_R\kappa_R}w_2=0\\ \frac{\delta w_2}{ds_R} &= w_1\varepsilon_R\sqrt{-2\varepsilon_R\kappa_R}+\frac{dw_2}{ds_R}=0.\end{aligned}\tag{2.16}$$

Substituting these equations into (2.15), we get

$$\begin{aligned}\frac{\delta w_1}{ds_B} &= (\sqrt{-2\varepsilon_R\kappa_R}-\sqrt{-2\varepsilon_B\kappa_B})w_2\\ \frac{\delta w_2}{ds_B} &= (\varepsilon_B\sqrt{-2\varepsilon_B\kappa_B}-\varepsilon_R\sqrt{-2\varepsilon_R\kappa_R})w_1.\end{aligned}\tag{2.17}$$

If we choose $\varepsilon_B=\varepsilon_R=-1$, then

$$0<(\frac{\delta w_1}{ds_B})^2-(\frac{\delta w_2}{ds_B})^2=(\sqrt{2\kappa_R}-\sqrt{2\kappa_B})^2(w_2^2-w_1^2).\tag{2.18}$$

Hence, we can put

$$w_1=r\sinh\phi\ ,\quad w_2=r\cosh\phi.$$

Differentiating this equations, we get

$$\begin{aligned}\frac{dw_1}{ds_R} &= \frac{dr}{ds_R}\sinh\phi+r\cosh\phi\frac{d\phi}{ds_R}\\ \frac{dw_2}{ds_R} &= \frac{dr}{ds_R}\cosh\phi+r\sinh\phi\frac{d\phi}{ds_R}\end{aligned}\tag{2.19}$$

Providing that we use (2.16), then we have

$$\begin{aligned}\frac{dw_1}{ds_R} &= r\sqrt{2\kappa_R}\cosh\phi-1\\ \frac{dw_2}{ds_R} &= r\sinh\phi\sqrt{2\kappa_R}\end{aligned}\tag{2.20}$$

If we consider (2.19) and (2.20), then we get

$$r\frac{d\phi}{ds_R}=-r\sqrt{2\kappa_R}+\cosh\phi\tag{2.21}$$

Therefore, substituting this equation into (2.14), we have

$$r\overline{\kappa}_L=\pm1+\frac{\cosh\phi}{r\left|\sqrt{2\kappa_R}-\sqrt{2\kappa_B}\right|}\tag{2.22}$$

If we choose $\varepsilon_B = \varepsilon_R = +1$, then from (2.17)

$$0 < (\frac{\delta w_1}{ds_B})^2 + (\frac{\delta w_2}{ds_B})^2 = (\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B})^2 (w_1^2 + w_2^2) \tag{2.23}$$

Hence we can put

$$w_1 = r\sin\phi \ , \quad w_2 = r\cos\phi.$$

Differentiating this equations, we get

$$\begin{aligned} \frac{dw_1}{ds_R} &= \frac{dr}{ds_R}\sin\phi + r\cos\phi\frac{d\phi}{ds_R} = r\sqrt{-2\kappa_R}\cos\phi - 1 \\ \frac{dw_2}{ds_R} &= \frac{dr}{ds_R}\cos\phi - r\sin\phi\frac{d\phi}{ds_R} = -r\sin\phi\sqrt{-2\kappa_R} \end{aligned} \tag{2.24}$$

and

$$r\frac{d\phi}{ds_R} = r\sqrt{-2\kappa_R} - \cos\phi \tag{2.25}$$

Therefore, substituting this equation into (2.14), we have

$$r\overline{\kappa}_L = \frac{\sqrt{-2\kappa_B} + \sqrt{-2\kappa_R}}{\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|} - \frac{\cos\phi}{r\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|}, \tag{2.26}$$

where $\overline{\kappa}_L = \sqrt{-2\varepsilon_L \kappa_L}$.

Thus we have the following Euler-Savary's Theorem for the planar curves in two dimensional lightlike cone Q^2.

Theorem 2.3 *Let x_R be a planar curve on the lightlike cone Q^2 such that it rolles without splitting along a curve x_B. Let x_L be a locus of a point P that is relative to x_R. Let Q be a point on x_L and R a point of contact of x_B and x_R corresponds to Q relative to the rolling relation. By (r, ϕ) , we denote a polar coordinate of Q with respect to the origin R and the base line $x_B^{'} |_R$. Then curvatures κ_B , κ_R and κ_L of x_B , x_R and x_L respectively, satisfies*

$$\begin{aligned} r\overline{\kappa}_L &= \pm 1 + \frac{\cosh\phi}{r\left|\sqrt{2\kappa_R} - \sqrt{2\kappa_B}\right|}, \quad \textit{if } \varepsilon_B = \varepsilon_R = -1, \\ r\overline{\kappa}_L &= \frac{\sqrt{-2\kappa_B} + \sqrt{-2\kappa_R}}{\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|} - \frac{\cos\phi}{r\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|} \quad \textit{if } \varepsilon_B = \varepsilon_R = +1. \end{aligned}$$

References

[1] Hano, J. and Nomizu, K., Surfaces of Revolution with Constant Mean Curvature in Lorentz-Minkowski Space, *Tohoku Math. J.*, 36 (1984), 427-437.

[2] Duggal, K.L. and Bejancu, A., *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers,1996.

[3] Liu, H., Curves in the Lightlike Cone, *Contributions to Algebra and Geometry*, 45(2004), No.1, 291-303.

[4] Ikawa, T., Euler-Savary's Formula on Minkowski Geometry, *Balkan Journal of Geometry and Its Applications*, 8(2003), No. 2, 31-36.

[5] Tosun, M., Güngör, M.A., Okur, I., On the 1-Parameter Lorentzian Spherical Motions and Euler-Savary formula, American Society of Mechanical Engineering, *Journal of Applied Mechanic*, vol.75, No.4, 972-977, September 2007.