ON THE MEAN VALUE OF THE $SCBF$ FUNCTION

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the SCBF function on simple numbers, and give an interesting asymptotic formula for it.

Keywords: SCBF function; Mean value; Asymptotic formula.

§1. Introduction

In reference [1], the Smarandache Sum of Composites Between Factors function $SCBF(n)$ is defined as: The sum of composite numbers between the smallest prime factor of n and the largest prime factor of n. For example, $SCBF(14)$=10, since $2\times 7 = 14$ and the sum of the composites between 2 and 7 is: $4 + 6 = 10$. In reference [2]: A number n is called simple number if the product of its proper divisors is less than or equal to n. Let A denotes set of all simple numbers. That is, $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \cdots\}$.

According to reference [1], Jason Earls has studied the arithmetical properties of $SCBF(n)$ and proved that $SCBF(n)$ is not a multiplicative function. For example, $SCBF(14 \times 15) = 10$ and $SCBF(14) \times SCBF(15) = 40$. He also got that if i and j are positive integers then $SCBF(2^i \times 5^j) = 4$, $SCBF(2^i \times 7^j) = 10$, etc. In this paper, we use the elementary method to study the mean value properties of $SCBF(n)$ on simple numbers, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let $x \geq 1$, A denotes the set of all simple numbers. Then we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in A}} SCBF(n) = B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

where $B = \frac{1}{3}\sum_{p}\frac{1}{p^3}$ is a constant, $\sum_{p}$ denotes the summation over all primes.

§2. Some Lemmas

To complete the proof of the theorem, we need the following lemmas:

Lemma 1. For any prime p and positive integer k, we have the asymptotic formula

$$SCBF(p^k) = 0.$$

Proof. (See reference [1]).

Lemma 2. Let $n \in A$, then we have $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$ four case, where p, q denote the distinct primes.

Proof. First let n be a positive integer, $p_d(n)$ is the product of all positive divisors of n, that is, $p_d(n) = \prod_{d|n} d$. $q_d(n)$ is the product of all positive divisors of n but n. That is, $q_d(n) = \prod_{d|n, d<n} d$. Then from the definition of $p_d(n)$ we know that

$$p_d(n) = \prod_{d|n} d = \prod_{d|n} \frac{n}{d}.$$

So from this formula we have

$$p_d^2(n) = \prod_{d|n} d \times \prod_{d|n} \frac{n}{d} = \prod_{d|n} n = n^{d(n)}.$$

where $d(n) = \sum_{d|n} 1$. Then we may immediately get $p_d(n) = n^{\frac{d(n)}{2}}$ and

$$q_d(n) = \prod_{d|n, d<n} d = \frac{\prod_{d|n} d}{n} = n^{\frac{d(n)}{2}-1}.$$

By the definition of the simple numbers, we get $n^{\frac{d(n)}{2}-1} \leq n$. Therefore, we have

$$d(n) \leq 4.$$

This inequality holds only for $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$ four cases. This completes the proof of Lemma 2.

Lemma 3. For any distinct prime p and q, we have the asymptotic formula

$$SCBF(pq) = \frac{q^2}{2}\left(1 - \frac{1}{\ln q}\right) - \frac{p^2}{2}\left(1 - \frac{1}{\ln p}\right) + O\left(\frac{q^2}{\ln^2 q}\right).$$

Proof. From the definition of $SCBF(n)$, we have

$$SCBF(pq) = \sum_{p<n<q} n - \sum_{p<q_1<q} q_1,$$

where q_1 is a prime. Using the Abel's Identity [3] and note that the asymptotic formula

$$\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha})$$

we can get

$$\begin{aligned}
SCBF(pq) &= \sum_{p<n<q} n - \sum_{p<q_1<q} q_1 \\
&= \sum_{p<n\le q-1} n - \sum_{p<q_1\le q-1} q_1 \\
&= \sum_{n\le q-1} n - \sum_{n\le p} n - \sum_{p<q_1\le q-1} q_1 \\
&= \frac{(q-1)^2}{2} - \frac{(p-1)^2}{2} + O(q) - (q-1)\pi(q-1) + p\pi(p) \\
&\quad + \int_p^{q-1} \pi(t)dt \\
&= \frac{q^2}{2} - \frac{q^2}{2\ln q} - \frac{p^2}{2} + \frac{p^2}{2\ln p} + O\left(\frac{q^2}{\ln^2 q}\right).
\end{aligned}$$

This completes the proof of Lemma 3.

Lemma 4. For real number $x \ge 1$, we have the asymptotic formula

$$\sum_{pq\le x} SCBF(pq) = B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right),$$

where p and q are two distinct primes, $B = \frac{1}{3}\sum_p \frac{1}{p^3}$ is a constant, and $\sum_p$ denotes the summation over all primes.

Proof. From the definition of $SCBF(n)$ and Lemma 1, Lemma 3, we get

$$\begin{aligned}
\sum_{pq\le x} SCBF(pq) &= 2\sum_{pq\le x, p<q} SCBF(pq) - \sum_{p^2\le x} SCBF(p^2) \\
&= 2\sum_{p\le\sqrt{x}} \sum_{p<q\le\frac{x}{p}} SCBF(pq) \\
&= \sum_{p\le\sqrt{x}} \sum_{p<q\le\frac{x}{p}} \left(q^2 - \frac{q^2}{\ln q} - p^2 + \frac{p^2}{\ln p} + O\left(\frac{q^2}{\ln^2 q}\right)\right).
\end{aligned}$$

Noting that $\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$, using Abel's Identity [3] we get

$$\begin{aligned}
\sum_{p<q\le\frac{x}{p}} q^2 &= \pi(\frac{x}{p})\frac{x^2}{p^2} - \pi(p)p^2 - 2\int_p^{\frac{x}{p}} \pi(t)tdt \\
&= \frac{x^3}{3p^3\ln\frac{x}{p}} - \frac{p^3}{3\ln p} + O\left(\frac{x^3}{p^3\ln^2\frac{x}{p}}\right)
\end{aligned}$$

and

$$\begin{aligned}\sum_{p<q\leq\frac{x}{p}}\frac{q^2}{\ln q} &= A(\frac{x}{p})f(\frac{x}{p})-A(p)f(p)-\int_p^{\frac{x}{p}}A(t)f(t)'dt\\ &= \frac{x^3}{3p^3\ln^2\frac{x}{p}}-\frac{p^3}{3\ln^2 p}-\frac{p^3}{9\ln^3 p}+O\left(\frac{x^3}{p^3\ln^3\frac{x}{p}}\right),\end{aligned}$$

where $A(\frac{x}{p})=\sum_{p<q\leq\frac{x}{p}}q^2$, $f(x)=\frac{1}{\ln x}$. From reference [3], we know that

$$\sum_{p\leq x}\frac{1}{p}=\ln\ln x+C+O\left(\frac{1}{\ln x}\right),$$

where C is a computable constant. And then we also get

$$\sum_{p\leq\sqrt{x}}p=\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right)$$

and

$$\sum_{p\leq\sqrt{x}}p^3=\frac{x^2}{2\ln x}+O\left(\frac{x^2}{\ln^2 x}\right).$$

Using the same method, we obtain

$$\sum_{p\leq\sqrt{x}}\frac{p}{\ln p}=\frac{2x}{\ln^2 x}+O\left(\frac{x}{\ln^3 x}\right)$$

and

$$\sum_{p\leq\sqrt{x}}\frac{p^3}{\ln p}=\frac{x^2}{\ln^2 x}+O\left(\frac{x^2}{\ln^3 x}\right).$$

Noting that $\frac{1}{1-\frac{\ln p}{\ln x}}=1+\frac{\ln p}{\ln x}+\frac{\ln^2 p}{\ln^2 x}+\cdots+\frac{\ln^m p}{\ln^m x}+\cdots$, then we get the following two formulae:

$$\begin{aligned}&\sum_{p\leq\sqrt{x}}\sum_{p<q\leq\frac{x}{p}}q^2\\ =&\sum_{p\leq\sqrt{x}}\left(\frac{x^3}{3p^3\ln\frac{x}{p}}-\frac{p^3}{3\ln p}+O\left(\frac{x^3}{p^3\ln^2\frac{x}{p}}\right)\right)\\ =&\frac{x^3}{3\ln x}\sum_{p\leq\sqrt{x}}\frac{1}{p^3}\left(1+\frac{\ln p}{\ln x}+\frac{\ln^2 p}{\ln^2 x}+\cdots\right)\\ &-\frac{1}{3}\sum_{p\leq\sqrt{x}}\frac{p^3}{\ln p}+O\left(\frac{x^3}{\ln^2 x}\sum_{p\leq\sqrt{x}}\frac{1}{p^3}\left(1+2\frac{\ln p}{\ln x}+3\frac{\ln^2 p}{\ln^2 x}+\cdots\right)\right)\\ =&C_1\frac{x^3}{\ln x}+O\left(\frac{x^3}{\ln^2 x}\right);\end{aligned}$$

$$\begin{aligned}
& \sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\frac{q^2}{\ln q}\\
= & \sum_{p\le\sqrt{x}}\left(\frac{x^3}{3p^3\ln^2\frac{x}{p}}-\frac{p^3}{3\ln^2 p}-\frac{p^3}{9\ln^3 p}+O\left(\frac{x^3}{p^3\ln^3\frac{x}{p}}\right)\right)\\
= & \frac{x^3}{3\ln^2 x}\sum_{p\le\sqrt{x}}\frac{1}{p^3}\left(1+2\frac{\ln p}{\ln x}+3\frac{\ln^2 p}{\ln^2 x}+\cdots\right)-\frac{1}{3}\sum_{p\le\sqrt{x}}\frac{p^3}{\ln^2 p}\\
& -\frac{1}{9}\sum_{p\le\sqrt{x}}\frac{p^3}{\ln^3 p}+O\left(\sum_{p\le\sqrt{x}}\frac{x^3}{p^3\ln^3\frac{x}{p}}\right)\\
= & C_2\frac{x^3}{\ln^2 x}+O\left(\frac{x^2}{\ln^2 x}\right),
\end{aligned}$$

where $C_1=C_2=\frac{1}{3}\sum\limits_p\frac{1}{p^3}$.

So we have

$$\begin{aligned}
& 2\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}SCBF(pq)\\
= & \sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\left(q^2-\frac{q^2}{\ln q}-p^2+\frac{p^2}{\ln p}+O\left(\frac{q^2}{\ln^2 q}\right)\right)\\
= & \sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}q^2-\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\frac{q^2}{\ln q}-\sum_{p\le\sqrt{x}}p^2\sum_{p<q\le\frac{x}{p}}1\\
& +\sum_{p\le\sqrt{x}}\frac{p^2}{\ln p}\sum_{p<q\le\frac{x}{p}}1+O\left(\sum_{p\le\sqrt{x}}\sum_{p<q\le\frac{x}{p}}\frac{q^2}{\ln^2 q}\right)\\
= & B\frac{x^3}{\ln x}+O\left(\frac{x^3}{\ln^2 x}\right),
\end{aligned}$$

where $B=\frac{1}{3}\sum\limits_p\frac{1}{p^3}$. This proves Lemma 4.

§3. Proof of the theorem

In this section, we complete the proof of Theorem. According to the definition of simple numbers and Lemma 2, we have

$$\sum_{\substack{n\le x\\ n\in A}}SCBF(n)$$

$$= \sum_{p\leq x} SCBF(p) + \sum_{p^2\leq x} SCBF(p^2) + \sum_{p^3\leq x} SCBF(p^3) + \sum_{pq\leq x} SCBF(pq).$$

And then, using Lemma 1 and Lemma 4 we obtain

$$\begin{aligned} \sum_{\substack{n\leq x\\ n\in A}} SCBF(n) &= \sum_{pq\leq x} SCBF(pq) \\ &= B\frac{x^3}{\ln x} + O\left(\frac{x^3}{\ln^2 x}\right). \end{aligned}$$

This completes the proof of Theorem.

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References

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