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On the mean value of $SSMP(n)$ and $SIMP(n)$[1]

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Abstract The main purpose of this paper it to studied the mean value properties of the Smarandache Superior m-th power part sequence $SSMP(n)$ and the Smarandache Inferior m-th power part sequence $SIMP(n)$, and give several interesting asymptotic formula for them.
Keywords Smarandache Superior m-th power part sequence, Smarandache Inferior m-th power part sequences, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the Smarandache Superior m-th power part sequence $SSMP(n)$ is defined as the smallest m-th power greater than or equal to n. The Smarandache Inferior m-th power part sequence $SIMP(n)$ is defined as the largest m-th power less than or equal to n. For example, if $m = 2$, then the first few terms of $SIMP(n)$ are: 0, 1, 1, 1, 4, 4, 4, 4, 4, 9, 9, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 16, 16, 25, $\cdots$. The first few terms of $SSMP(n)$ are: 1, 4, 4, 4, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 25, $\cdots$. If $m = 3$, then The first few terms of $SSMP(n)$ are: 1, 8, 8, 8, 8, 8, 8, 8, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 64, $\cdots$. The first few terms of $SIMP(n)$ are: 0, 1, 1, 1, 1, 1, 1, 1, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 27, $\cdots$. Now we let

$$\begin{aligned} S_n &= (SSMP(1) + SSMP(2) + \cdots + SSMP(n))/n; \\ I_n &= (SIMP(1) + SIMP(2) + \cdots + SIMP(n))/n; \\ K_n &= \sqrt[n]{SSMP(1) + SSMP(2) + \cdots + SSMP(n)}; \\ I_n &= \sqrt[n]{SIMP(1) + SIMP(2) + \cdots + SIMP(n)}. \end{aligned}$$

In reference [2], Dr. K.Kashihara asked us to study the properties of these sequences. Gou Su [3] studied these problem, and proved the following conclusion:

For any real number $x > 2$ and integer $m = 2$, we have the asymptotic formula

$$\sum_{n \leqslant x} SSSP\,(n) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right), \quad \sum_{n \leqslant x} SISP\,(n) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right),$$

and

$$\tfrac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{2}}\right), \quad \lim_{n \to \infty} \tfrac{S_n}{I_n} = 1.$$

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In this paper, we shall use the elementary method to give a general conclusion. That is, we shall prove the following:

Theorem 1. Let $m \geq 2$ be an integer, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right),$$

and

$$\sum_{n \leq x} SIMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right).$$

Theorem 2. For any fixed positive integer $m \geq 2$ and any positive integer n, we have the asymptotic formula

$$S_n - I_n = \frac{m(m-1)}{2m-1} n^{1-\frac{1}{m}} + O\left(n^{1-\frac{2}{m}}\right).$$

Corollary 1. For any positive integer n, we have the asymptotic formula

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{m}}\right),$$

and the limit $\lim_{n\to\infty} \frac{S_n}{I_n} = 1$.

Corollary 2. For any positive integer n, we have the asymptotic formula

$$\frac{K_n}{L_n} = 1 + O\left(\frac{1}{n}\right),$$

and the limit $\lim_{n\to\infty} \frac{K_n}{L_n} = 1$, $\lim_{n\to\infty} (K_n - L_n) = 0$.

§2. Proof of the theorems

In this section, we shall use the Euler summation formula and the elementary method to complete the proof of our Theorems. For any real number $x > 2$, it is clear that there exists one and only one positive integer M satisfying $M^m < x \leq (M+1)^m$. That is, $M = x^{\frac{1}{m}} + O(1)$. So we have

$$\begin{aligned}
\sum_{n \leq x} SSMP(n) &= \sum_{n \leq M^m} SSMP(n) + \sum_{M^m < n \leq x} SSMP(n) \\
&= \sum_{k \leq M} (k^m - (k-1)^m) k^m + ([x] - (M^m + 1))(M+1)^m \\
&= \sum_{k \leq M} (m k^{2m-1} + O(k^{2m-2})) + ([x] - M^m - 1)(M+1)^m \\
&= \frac{m \cdot M^{2m}}{2m} + O\left(M^{2m-1}\right) + ([x] - M^m - 1)(M+1)^m \\
&= \frac{M^{2m}}{2} + O\left(M^{2m-1}\right).
\end{aligned}$$

Note that $M = x^{\frac{1}{m}} + O(1)$, from the above estimate we have the asymptotic formula

$$\sum_{n\leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2-\frac{1}{m}}\right).$$

This proves the first formula of Theorem 1.

Now we prove the second one. For any real number $x > 1$, we also have

$$\begin{aligned}\sum_{n\leq x} SIMP(n) &= \sum_{n<M^m} SIMP(n) + \sum_{M^m\leq n\leq x} SIMP(n)\\ &= \sum_{k\leqslant M}(k^m-(k-1^m))(k-1)^m + \sum_{M^m\leqslant n\leqslant x} M^m\\ &= \sum_{k\leqslant M}(mk^{2m-1}+O(k^{2m-2})) + ([x]-M^m+1)\,M^m\\ &= \frac{M^{2m}}{2} + O\left(M^{2m-1}\right) + ([x]-M^m+1)\,M^m.\end{aligned}$$

Note that

$$([x]-M^m+1)\,M^m \leqslant M^{2m-1} \leq x^{1-\frac{1}{m}}.$$

Therefore,

$$\sum_{n\leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2-\frac{1}{m}}\right).$$

This completes the proof of Theorem 1.

To prove Theorem 2, let $x = n$, then from the method of proving Theorem 1 we have

$$\begin{aligned}S_n - I_n &= \frac{1}{n}\left(SSMP(1)+SSMP(2)+\cdots+SSMP(n)\right)\\ &\quad -\frac{1}{n}\left(SIMP(1)+SIMP(2)+\cdots+SIMP(n)\right)\\ &= \frac{1}{n}\left(\sum_{k\leq M}(k^m-(k-1)^m)k^m + ([x]-(M^m+1))(M+1)^m\right)\\ &\quad -\frac{1}{n}\left(\sum_{k\leqslant M}(k^m-(k-1^m))(k-1)^m + ([x]-M^m+1)\,M^m\right)\\ &= \frac{1}{n}\sum_{k\leq M} m(m-1)k^{2m-2} + O\left(\frac{1}{n}M^{2m-2}\right)\\ &= \frac{m(m-1)}{n(2m-1)}M^{2m-1} + O\left(\frac{1}{n}M^{2m-2}\right).\end{aligned}$$

Note that $M^m < n \leq (M+1)^m$ or $M = n^{\frac{1}{m}} + O(1)$, from the above formula we may immediately deduce that

$$S_n - I_n = \frac{m(m-1)}{2m-1}n^{1-\frac{1}{m}} + O\left(n^{1-\frac{2}{m}}\right).$$

This completes the proof of Theorem 2.

Now we prove the Corollaries. Note that the asymptotic formula

$$I_n = \frac{1}{n}(SIMP(1) + SIMP(2) + \cdots + SIMP(n)) = \frac{1}{n}\left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right) = \frac{n}{2} + O\left(n^{1-\frac{1}{m}}\right)$$

and

$$S_n = \frac{1}{n}\left(SSMP(1) + SSMP(2) + \cdots + SSMP(n)\right) = \frac{1}{n}\left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right) = \frac{n}{2} + O\left(n^{1-\frac{1}{m}}\right).$$

From the above two formula we have

$$\frac{S_n}{I_n} = \frac{\frac{n}{2} + O\left(n^{\frac{m-1}{m}}\right)}{\frac{n}{2} + O\left(n^{\frac{m-1}{m}}\right)} = 1 + O\left(n^{-\frac{1}{m}}\right).$$

Therefore, we have the limit formula

$$\lim_{n\to\infty} \frac{S_n}{I_n} = 1.$$

Using the same method we can also deduce that

$$K_n = \sqrt[n]{SSMP(1) + SSMP(2) + \cdots + SSMP(n)} = \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right)^{\frac{1}{n}}$$

and

$$L_n = \sqrt[n]{SIMP(1) + SIMP(2) + \cdots + SIMP(n)} = \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)\right)^{\frac{1}{n}}$$

From these formula we may immediately deduce that

$$\frac{K_n}{L_n} = \left(\frac{\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)}{\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)}\right)^{\frac{1}{n}} = \left(1 + O\left(n^{-\frac{1}{m}}\right)\right)^{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right).$$

Therefore, we have the limit formula

$$\lim_{n\to\infty} \frac{K_n}{L_n} = 1.$$

Note that $\lim_{n\to\infty} K_n = \lim_{n\to\infty} L_n = 1$, we may immediately deduce that

$$\lim_{n\to\infty} (K_n - L_n) = 0.$$

This completes the proof of Corollary 2.

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