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A new critical method for twin primes

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Abstract For any positive integer $n \geq 3$, if n and $n+2$ both are primes, then we call that n and $n+2$ are twin primes. In this paper, we using the elementary method to study the relationship between the twin primes and some arithmetical function, and give a new critical method for twin primes.

Keywords The Smarandache reciprocal function, critical method for twin primes.

§1. Introduction and result

For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$, and $m+1 \dagger n!$. That is, $S_c(n) = \max\{m : \ y \mid n! \text{ for all } 1 \leq y \leq m, \text{ and } \ m+1 \dagger n!\}$. From the definition of $S_c(n)$ we can easily deduce that the first few values of $S_c(n)$ are:

$$S_c(1)=1,\ S_c(2)=2,\ S_c(3)=3,\ S_c(4)=4,\ S_c(5)=6,\ S_c(6)=6,$$
$$S_c(7)=10,\ S_c(8)=10,\ S_c(9)=10,\ S_c(10)=10,\ S_c(11)=12,\ S_c(12)=12,$$
$$S_c(13)=16,\ S_c(14)=16,\ S_5(15)=16,\ S_c(16)=16,\ S_c(17)=18,\ \cdots\cdots.$$

About the elementary properties of $S_c(n)$, many authors had studied it, and obtained a series results, see references [2], [3] and [4]. For example, A.Murthy [2] proved the following conclusion:

If $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

Ding Liping [3] proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} S_c(n) = \frac{1}{2} \cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

On the other hand, Jozsef Sandor [5] introduced another arithmetical function $P(n)$ as follows: $P(n) = \min\{p : \ n \,|p!, \text{ where } p \text{ be a prime}\}$. That is, $P(n)$ denotes the smallest prime p such that $n \mid p!$. In fact function $P(n)$ is a generalization of the Smarandache function $S(n)$. Its some values are: $P(1)=2$, $P(2)=2$, $P(3)=3$, $P(4)=5$, $P(5)=5$, $P(6)=3$, $P(7)=7$, $P(8)=5$, $P(9)=7$, $P(10)=5$, $P(11)=11$, $\cdots$. It is easy to prove that for each prime p one has $P(p)=p$, and if n is a square-free number, then $P(n)$ = greatest prime divisor of n. If p be a prime, then the following double inequality is true:

$$2p+1 \leq P(p^2) \leq 3p-1 \tag{1}$$

and

$$S(n) \leq P(n) \leq 2S(n) - 1. \qquad (2)$$

In reference [6], Li Hailong studied the value distribution properties of $P(n)$, and proved that for any real number $x > 1$, we have the mean value formula

$$\sum_{n \leq x} \left(P(n) - \overline{P}(n)\right)^2 = \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\overline{P}(n)$ denotes the largest prime divisor of n, and $\zeta(s)$ is the Riemann zeta-function.

In this paper, we using the elementary method to study the solvability of an equation involving the Smarandache reciprocal function $S_c(n)$ and $P(n)$, and give a new critical method for twin primes. That is, we shall prove the following:

Theorem. For any positive integer $n > 3$, n and $n + 2$ are twin primes if and only if n satisfy the equation

$$S_c(n) = P(n) + 1. \qquad (3)$$

§2. Proof of the theorem

In this section, we shall prove our theorem directly. First we prove that if n (> 3) and $n + 2$ both are primes, then n satisfy the equation (3). In fact this time, from A.Murthy [2] we know that $S_c(n) = n + 1$ and $P(n) = n$, so $S_c(n) = P(n) + 1$, and n satisfy the equation (3).

Now we prove that if $n > 3$ satisfy the equation $S_c(n) = P(n) + 1$, then n and $n + 2$ both are primes. We consider n in following three cases:

(A) If $n = q$ be a prime, then $P(n) = P(q) = q$, and $S_c(q) = P(q) + 1 = q + 1$, note that $q > 3$, so from [2] we know that $q + 2$ must be a prime. Thus n and $n + 2$ both are primes.

(B) If $n = q^\alpha$, q be a prime and $\alpha \geq 2$, then from the estimate (2) and the properties of the Smarandache function $S(n)$ we have

$$P\left(q^\alpha\right) \leq 2S\left(q^\alpha\right) - 1 \leq 2\alpha q - 1.$$

On the other hand, from [2] we also have

$$S_c\left(q^\alpha\right) \geq q^\alpha + 2, \text{ if } q \geq 3; \quad \text{and} \quad S_c\left(2^\alpha\right) \geq 2^\alpha + 1.$$

If $S_c\left(q^\alpha\right) = P\left(q^\alpha\right) + 1$, then from the above two estimates we have the inequalities

$$q^\alpha + 3 \leq 2\alpha q \qquad (4)$$

and

$$2^\alpha + 2 \leq 4\alpha. \qquad (5)$$

It is clear that (4) does not hold if $q \geq 5$ $(q = 3)$ and $\alpha \geq 2$ $(\alpha \geq 3)$. If $n = 3^2$, then $S_c(9) = 10$, $P(9) = 7$, so we also have $S_c(9) \neq P(9) + 1$.

It is easy to check that the inequality (5) does not hold if $\alpha \geq 4$. $S_c(2) \neq P(2)+1$, $S_c(4) \neq P(4)+1$, $S_c(8) \neq P(8)+1$.

Therefore, if $n = q^\alpha$, where q be a prime and $\alpha \geq 2$ be an integer, then n does not satisfy the equation (3).

(C) If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $k \geq 2$ be an integer, p_i $(i = 1, 2, \cdots, k)$ are primes, and $\alpha_i \geq 1$. From the definition of $S_c(n)$ and the inequality (2) we have $S_c(n) \geq n$ and

$$P(n) \leq 2S(n) - 1 = 2 \cdot \max_{1\leq i\leq k}\{S(p_i^{\alpha_i})\} - 1 \leq 2 \cdot \max_{1\leq i\leq k}\{\alpha_i p_i\} - 1.$$

So if n satisfy the equation (3), then we have

$$n \leq S_c(n) = P(n) + 1 \leq 2 \cdot S(n) \leq 2 \cdot \max_{1\leq i\leq k}\{\alpha_i p_i\}.$$

Let $\max\limits_{1\leq i\leq k}\{\alpha_i p_i\} = \alpha \cdot p$ and $n = p^\alpha \cdot n_1$, $n_1 > 1$. Then from the above estimate we have

$$p^\alpha \cdot n_1 \leq 2 \cdot \alpha \cdot p. \tag{6}$$

Note that n has at least two prime divisors, so $n_1 \geq 2$, thus (6) does not hold if $p \geq 3$ and $\alpha > 1$. If $p = 2$, then $n_1 \geq 3$. In any case, n does not satisfy the equation (3).

This completes the proof of Theorem.

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