

# A NEW FUNCTION AND ITS MEAN VALUE\*

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**Abstract** The main purpose of this paper is using the elementary method to study the mean value properties of a new function for  $n$ , and give a sharp asymptotic formula for it.

**Keywords:** Elementary method; Mean value; Asymptotic formula.

## §1. Introduction

For any positive integer  $n$ , let  $Sg(n)$  denotes the smallest square greater than or equal to  $n$ . For example,  $Sg(1) = 1$ ,  $Sg(2) = 4$ ,  $Sg(3) = 4$ ,  $Sg(4) = 4$ ,  $Sg(5) = 9$ ,  $Sg(6) = 9$ ,  $Sg(7) = 9$ ,  $\dots$ ,  $Sg(9) = 9$ ,  $Sg(10) = 16 \dots$ . In problem 40 of book [1], Professor F. Smarandache asks us to study the properties of the sequence  $Sg(n)$ . About this problem, we know very little. Let  $x$  be any real number, in this paper we will study function  $Sk(x) - x$ . The problem is very important because it can help us to study the distribution of the square root sequence. In this paper, we generalized this problem for generalization. That is, let  $Sk(n)$  denotes the smallest power  $k$  greater than or equal to  $n$ , and  $Gk(n) = Sk(n) - n$ . In this paper use the elementary methods to study the mean value properties of  $Gk(n)$  for  $n$ , and give a sharp asymptotic formula for it. That is, we shall prove the following:

**Theorem.** Let  $x \geq 1$ , we have the asymptotic formula

$$\sum_{n \leq x} Gk(n) = \frac{k^2}{2(2k-1)} x^{\frac{2k-1}{k}} + O\left(x^{\frac{2k-2}{k}}\right).$$

Especially, when  $k = 2, 3$ , we have the following

**Corollary 1.** Let  $x \geq 1$ , we have

$$\sum_{n \leq x} (Sg(n) - n) = \frac{2}{3} x^{\frac{3}{2}} + O(x).$$

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**Corollary 2.** Let  $x \geq 1$  and  $Sc(n)$  denotes the smallest cube greater than or equal to  $n$ , we have

$$\sum_{n \leq x} (Sc(n) - n) = \frac{9}{10} x^{\frac{5}{3}} + O\left(x^{\frac{4}{3}}\right).$$

## §2. A Lemma

To complete the proof of the theorem, we need the following:

**Lemma.** Let  $x$  be any real number and  $\alpha \geq 0$ , we have

$$\sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + o(x^\alpha).$$

**Proof** (See reference [2]).

## §3. Proof of the theorem

In this section, we complete the proof of Theorem. For any real number  $x \geq 1$ , let  $M$  be a fixed positive integer such that

$$M^k \leq x < (M+1)^k.$$

Then from the definition of  $Gk(n)$ , we have

$$\begin{aligned} \sum_{n \leq x} Gk(n) &= \sum_{t=1}^M \sum_{(t-1)^k \leq n < t^k} (t^k - n) + O\left(\sum_{M^k \leq n < x} (M^k - n)\right) \\ &= \sum_{t=1}^M \sum_{0 \leq u < t^k - (t-1)^k} u + O\left(\sum_{0 \leq n < M^k - x} u\right) \\ &= \sum_{t=1}^M \left[ \frac{k^2 t^{2k-2}}{2} + O(t^{2k-3}) \right] + O(M^{2k-2}) \\ &= \frac{k^2}{2(2k-1)} M^{2k-1} + O(M^{2k-2}) \end{aligned}$$

On the other hand, note that the estimates

$$0 \leq x - M^k \ll (M+1)^k - M^k \ll M^{k-1} \ll x^{\frac{k-1}{k}}$$

then we have

$$M^{2k-1} = x^{\frac{2k-1}{k}} + O\left(x^{\frac{2k-2}{k}}\right)$$

and

$$M^{2k-2} \ll x^{\frac{2k-2}{k}}$$

Now combining the above, we have obtain the asymptotic formula

$$\sum_{n \leq x} Gk(n) = \frac{k^2}{2(2k-1)} x^{\frac{2k-1}{k}} + O\left(x^{\frac{2k-2}{k}}\right).$$

This completes the proof of the theorem.

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### **References**

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