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Non-Solvable Spaces of Linear Equation Systems

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Abstract: A *Smarandache system* $(\Sigma;\mathcal{R})$ is such a mathematical system that has at least one *Smarandachely denied rule* in $\mathcal{R}$, i.e., there is a rule in $(\Sigma;\mathcal{R})$ that behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways. For such systems, the linear equation systems without solutions, i.e., non-solvable linear equation systems are the most simple one. We characterize such non-solvable linear equation systems with their homeomorphisms, particularly, the non-solvable linear equation systems with 2 or 3 variables by combinatorics. It is very interesting that every planar graph with each edge a straight segment is homologous to such a non-solvable linear equation with 2 variables.

Key Words: Smarandachely denied axiom, Smarandache system, non-solvable linear equations, $\vee$-solution, $\wedge$-solution.

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§1. Introduction

Finding the exact solution of equation system is a main but a difficult objective unless the case of linear equations in classical mathematics. Contrary to this fact, *what is about the non-solvable case?* In fact, such an equation system is nothing but a contradictory system, and characterized only by non-solvable equations for conclusion. But our world is overlap and hybrid. The number of non-solvable equations is more than that of the solvable. The main purpose of this paper is to characterize the behavior of such linear equation systems.

Let $\mathbb{R}^m$, $\mathbb{R}^m$ be Euclidean spaces with dimensional $m,\ n \geq 1$ and $T:\ \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a $\mathbb{C}^k,\ 1 \leq k \leq \infty$ function such that $T(\overline{x}_0, \overline{y}_0) = \overline{0}$ for $\overline{x}_0 \in \mathbb{R}^n$, $\overline{y}_0 \in \mathbb{R}^m$ and the $m \times m$ matrix $\partial T^j/\partial y^i(\overline{x}_0, \overline{y}_0)$ is non-singular, i.e.,

$$\det(\frac{\partial T^j}{\partial y^i})\left|_{(\overline{x}_0,\overline{y}_0)}\right. \neq 0,\ \text{where} 1 \leq i, j \leq m.$$

Then the implicit function theorem ([1]) implies that there exist opened neighborhoods $V \subset \mathbb{R}^n$ of $\overline{x}_0$, $W \subset \mathbb{R}^m$ of $\overline{y}_0$ and a $\mathbb{C}^k$ function $\phi: V \rightarrow W$ such that

$$T(\overline{x}, \phi(\overline{x})) = \overline{0}.$$

Thus there always exists solutions for the equation $T(\overline{x}, \overline{(y)}) = \overline{0}$ if T is $\mathbb{C}^k,\ 1 \leq k \leq \infty$. Now let $T_1, T_2, \cdots, T_m,\ m \geq 1$ be different $\mathbb{C}^k$ functions $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ for an integer $k \geq 1$. An

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equation system discussed in this paper is with the form following

$$T_i(\overline{x},\overline{y})=\overline{0},\quad 1\leq i\leq m. \qquad (Eq)$$

A point $(\overline{x}_0,\overline{y}_0)$ is a $\vee$-*solution* of the equation system (Eq) if

$$T_i(\overline{x}_0,\overline{y}_0)=\overline{0}$$

for some integers $i,\ 1\leq i\leq m$, and a $\wedge$-*solution* of (Eq) if

$$T_i(\overline{x}_0,\overline{y}_0)=\overline{0}$$

for all integers $1\leq i_0\leq m$. Denoted by S_i^0 the solutions of equation $T_i(\overline{x},\overline{y})=\overline{0}$ for integers $1\leq i\leq m$. Then $\bigcup\limits_{i=1}^{m}S_i^0$ and $\bigcap\limits_{i=1}^{m}S_i^0$ are respectively the $\vee$-solutions and $\wedge$-solutions of equations (Eq). By definition, we are easily knowing that the $\wedge$-solution is nothing but the same as the classical solution.

Definition 1.1 *The $\vee$-solvable, $\wedge$-solvable and non-solvable spaces of equations (Eq) are respectively defined by*

$$\bigcup_{i=1}^{m}S_i^0,\quad \bigcap_{i=1}^{m}S_i^0\quad and\quad \bigcup_{i=1}^{m}S_i^0-\bigcap_{i=1}^{m}S_i^0.$$

Now we construct a finite graph $G[Eq]$ of equations (Eq) following:

$V(G[Eq])=\{v_i|1\leq i\leq m\}$,

$E(G[Eq])=\{(v_i,v_j)|\exists(\overline{x}_0,\overline{y}_0)\Rightarrow T_i(\overline{x}_0,\overline{y}_0)=\overline{0}\wedge T_j(\overline{x}_0,\overline{y}_0)=\overline{0},\ 1\leq i,j\leq m\}$.

Such a graph $G[Eq]$ can be also represented by a vertex-edge labeled graph $G^L[Eq]$ following:

$V(G^L[Eq])=\{S_i^0|1\leq i\leq m\}$,

$E(G[Eq])=\{(S_i^0,S_j^0)$ labeled with $S_i^0\bigcap S_j^0|S_i^0\bigcap S_j^0\neq\emptyset,1\leq i,j\leq m\}$.

For example, let $S_1^0=\{a,b,c\},\ S_2^0=\{c,d,e\},\ S_3^0=\{a,c,e\}$ and $S_4^0=\{d,e,f\}$. Then its edge-labeled graph $G[Eq]$ is shown in Fig.1 following.

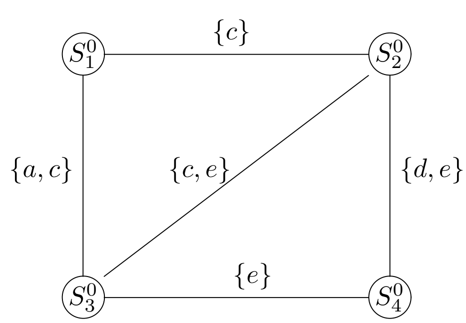

Fig.1

Notice that $\bigcup_{i=1}^{m} S_i^0 = \bigcup_{i=1}^{m} S_i^0$, i.e., the non-solvable space is empty only if $m = 1$ in (Eq). Generally, let $(\Sigma_1; \mathcal{R}_1)$ $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be mathematical systems, where $\mathcal{R}_i$ is a rule on Σ_i for integers $1 \le i \le m$. If for two integers i, j, $1 \le i, j \le m$, $\Sigma_i \ne \Sigma_j$ or $\Sigma_i = \Sigma_j$ but $\mathcal{R}_i \ne \mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*.

Definition 1.2([12]-[13]) *A rule in $\mathcal{R}$ a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$.

Thus, such a Smarandache system is a contradictory system. Generally, we know the conception of Smarandache multi-space with its underlying combinatorial structure defined following.

Definition 1.3([8]-[10]) *Let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be $m \ge 2$ mathematical spaces, different two by two. A Smarandache multi-space $\widetilde{\Sigma}$ is a union $\bigcup_{i=1}^{m} \Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i$ on $\widetilde{\Sigma}$, i.e., the rule $\mathcal{R}_i$ on Σ_i for integers $1 \le i \le m$, denoted by $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$.*

Similarly, the underlying graph of a Smarandache multi-space $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ is an edge-labeled graph defined following.

Definition 1.4([8]-[10]) *Let $\left(\widetilde{\Sigma}; \widetilde{\mathcal{R}}\right)$ be a Smarandache multi-space with $\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i$ and $\widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i$. Its underlying graph $G\left[\widetilde{\Sigma}, \widetilde{R}\right]$ is defined by*

$$V\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \ne \emptyset, 1 \le i, j \le m\}$$

with an edge labeling

$$l^E:\ (\Sigma_i, \Sigma_j) \in E\left(G\left[\widetilde{S}, \widetilde{R}\right]\right) \to l^E(\Sigma_i, \Sigma_j) = \varpi\left(\Sigma_i \bigcap \Sigma_j\right),$$

where ϖ is a characteristic on $\Sigma_i \bigcap \Sigma_j$ such that $\Sigma_i \bigcap \Sigma_j$ is isomorphic to $\Sigma_k \bigcap \Sigma_l$ if and only if $\varpi(\Sigma_i \bigcap \Sigma_j) = \varpi\left(\Sigma_k \bigcap \Sigma_l\right)$ for integers $1 \le i, j, k, l \le m$.

We consider the simplest case, i.e., all equations in (Eq) are linear with integers $m \ge n$ and $m, n \ge 1$ in this paper because we are easily know the necessary and sufficient condition of a linear equation system is solvable or not in linear algebra. For terminologies and notations not mentioned here, we follow [2]-[3] for linear algebra, [8] and [10] for graphs and topology.

Let

$$AX = (b_1, b_2, \cdots, b_m)^T \qquad (LEq)$$

be a linear equation system with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

for integers $m,\ n \geq 1$. Define an augmented matrix A^+ of A by $(b_1, b_2, \cdots, b_m)^T$ following:

$$A^+ = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

We assume that all equations in (LEq) are non-trivial, i.e., there are no numbers λ such that

$$(a_{i1}, a_{i2}, \cdots, a_{in}, b_i) = \lambda(a_{j1}, a_{j2}, \cdots, a_{jn}, b_j)$$

for any integers $1 \leq i, j \leq m$. Such a linear equation system (LEq) is *non-solvable* if there are no solutions $x_i,\ 1 \leq i \leq n$ satisfying (LEq).

§2. A Necessary and Sufficient Condition for Non-Solvable Linear Equations

The following result on non-solvable linear equations is well-known in linear algebra([2]-[3]).

Theorem 2.1 *The linear equation system (LEq) is solvable if and only if* $\operatorname{rank}(A) = \operatorname{rank}(A^+)$. *Thus, the equation system (LEq) is non-solvable if and only if* $\operatorname{rank}(A) \neq \operatorname{rank}(A^+)$.

We introduce the conception of parallel linear equations following.

Definition 2.2 *For any integers* $1 \leq i, j \leq m,\ i \neq j$, *the linear equations*

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i,$$
$$a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j$$

are called parallel if there exists a constant c *such that*

$$c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in} \neq b_j/b_i.$$

Then we know the following conclusion by Theorem 2.1.

Corollary 2.3 *For any integers* $i, j,\ i \neq j$, *the linear equation system*

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j \end{cases}$$

is non-solvable if and only if they are parallel.

Proof By Theorem 2.1, we know that the linear equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i,$$
$$a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j$$

is non-solvable if and only if $\mathrm{rank}A' \neq \mathrm{rank}B'$, where

$$A' = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix}, \quad B' = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} & b_1 \\ a_{j1} & a_{j2} & \cdots & a_{jn} & b_2 \end{bmatrix}.$$

It is clear that $1 \leq \mathrm{rank}A' \leq \mathrm{rank}B' \leq 2$ by the definition of matrixes A' and B'. Consequently, $\mathrm{rank}A' = 1$ and $\mathrm{rank}B' = 2$. Thus the matrix $A',\ B'$ are respectively elementary equivalent to matrixes

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix}.$$

i.e., there exists a constant c such that $c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in}$ but $c \neq b_j/b_i$. Whence, the linear equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i,$$
$$a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j$$

is parallel by definition. □

We are easily getting another necessary and sufficient condition for non-solvable linear equations (LEq) by three elementary transformations on a $m \times (n+1)$ matrix A^+ defined following:

(1) *Multiplying one row of A^+ by a non-zero scalar c;*

(2) *Replacing the ith row of A^+ by row i plus a non-zero scalar c times row j;*

(3) *Interchange of two row of A^+.*

Such a transformation naturally induces a transformation of linear equation system (LEq), denoted by $T(LEq)$. By applying Theorem 2.1, we get a generalization of Corollary 2.3 for non-solvable linear equation system (LEq) following.

Theorem 2.4 *A linear equation system (LEq) is non-solvable if and only if there exists a composition T of series elementary transformations on A^+ with $T(A^+)$ the forms following*

$$T(A^+) = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a'_{21} & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} & b'_m \end{bmatrix}$$

and integers $i,\ j$ with $1 \leq i,j \leq m$ such that the equations

$$a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n = b'_i,$$
$$a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n = b'_j$$

are parallel.

Proof Notice that the solution of linear equation system following

$$T(A)X = (b'_1, b'_2, \cdots, b'_m)^T \qquad (LEq^*)$$

has exactly the same solution with (LEq). If there are indeed integers k and $i,\ j$ with $1 \leq k,i,j \leq m$ such that the equations

$$a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n = b'_i,$$
$$a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n = b'_j$$

are parallel, then the linear equation system (LEq^*) is non-solvable. Consequently, the linear equation system (LEq) is also non-solvable.

Conversely, if for any integers k and $i,\ j$ with $1 \leq k,i,j \leq m$ the equations

$$a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n = b'_i,$$
$$a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n = b'_j$$

are not parallel for any composition T of elementary transformations, then we can finally get a linear equation system

$$\begin{cases} x_{l_1} + c_{1,s+1}x_{l_{s+1}} + \cdots + c_{1,n}x_{l_n} = d_1 \\ x_{l_2} + c_{2,s+1}x_{l_{s+1}} + \cdots + c_{2,n}x_{l_n} = d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{l_s} + c_{s,s+1}x_{l_{s+1}} + \cdots + c_{s,n}x_{l_n} = d_s \end{cases} \qquad (LEq^{**})$$

by applying elementary transformations on (LEq) from the knowledge of linear algebra, which has exactly the same solution with (LEq). But it is clear that (LEq^{**}) is solvable, i.e., the linear equation system (LEq) is solvable. Contradicts to the assumption. □

This result naturally determines the combinatorial structure underlying a linear equation system following.

Theorem 2.5 *A linear equation system (LEq) is non-solvable if and only if there exists a composition T of series elementary transformations such that*

$$G[T(LEq)] \not\simeq K_m,$$

where K_m is a complete graph of order m.

Proof Let $T(A^+)$ be

$$T(A^+) = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ a'_{21} & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \cdots & \cdots & \cdots & \cdots & \\ a'_{m1} & a'_{m2} & \cdots & a'_{mn} & b'_m \end{bmatrix}.$$

If there are integers $1 \leq i, j \leq m$ such that the linear equations

$$a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n = b'_i,$$
$$a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n = b'_j$$

are parallel, then there must be $S_i^0 \bigcap S_j^0 = \emptyset$, where $S_i^0,\ S_j^0$ are respectively the solutions of linear equations $a'_{i1}x_1 + a'_{i2}x_2 + \cdots a'_{in}x_n = b'_i$ and $a'_{j1}x_1 + a'_{j2}x_2 + \cdots a'_{jn}x_n = b'_j$. Whence, there are no edges (S_i^0, S_j^0) in $G[LEq]$ by definition. Thus $G[LEq] \not\simeq K_m$. $\square$

We wish to find conditions for non-solvable linear equation systems (LEq) without elementary transformations. In fact, we are easily determining $G[LEq]$ of a linear equation system (LEq) by Corollary 2.3. Let L_i be the ith linear equation. By Corollary 2.3, we divide these equations $L_i,\ 1 \leq i \leq m$ into *parallel families*

$$\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$$

by the property that all equations in a family $\mathscr{C}_i$ are parallel and there are no other equations parallel to lines in $\mathscr{C}_i$ for integers $1 \leq i \leq s$. Denoted by $|\mathscr{C}_i| = n_i,\ 1 \leq i \leq s$. Then the following conclusion is clear by definition.

Theorem 2.6 *Let (LEq) be a linear equation system for integers $m, n \geq 1$. Then*

$$G[LEq] \simeq K_{n_1, n_2, \cdots, n_s}$$

with $n_1 + n + 2 + \cdots + n_s = m$, where $\mathscr{C}_i$ is the parallel family with $n_i = |\mathscr{C}_i|$ for integers $1 \leq i \leq s$ in (LEq) and (LEq) is non-solvable if $s \geq 2$.

Proof Notice that equations in a family $\mathscr{C}_i$ is parallel for an integer $1 \leq i \leq m$ and each of them is not parallel with all equations in $\bigcup\limits_{1 \leq l \leq m, l \neq i} \mathscr{C}_l$. Let $n_i = |\mathscr{C}_i|$ for integers $1 \leq i \leq s$ in (LEq). By definition, we know

$$G[LEq] \simeq K_{n_1, n_2, \cdots, n_s}$$

with $n_1 + n + 2 + \cdots + n_s = m$.

Notice that the linear equation system (LEq) is solvable only if $G[LEq] \simeq K_m$ by definition. Thus the linear equation system (LEq) is non-solvable if $s \geq 2$. $\square$

Notice that the conditions in Theorem 2.6 is not sufficient, i.e., if $G[LEq] \simeq K_{n_1, n_2, \cdots, n_s}$, we can not claim that (LEq) is non-solvable or not. For example, let (LEq^*) and (LEq^{**}) be two linear equations systems with

$$A_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \qquad A_2^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} \qquad .$$

Then $G[LEq^*] \simeq G[LEq^{**}] \simeq K_4$. Clearly, the linear equation system (LEq^*) is solvable with $x_1 = 0$, $x_2 = 0$ but (LEq^{**}) is non-solvable. We will find necessary and sufficient conditions for linear equation systems with two or three variables just by their combinatorial structures in the following sections.

§3. Linear Equation System with 2 Variables

Let

$$AX = (b_1, b_2, \cdots, b_m)^T \qquad (LEq2)$$

be a linear equation system in 2 variables with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \cdots & \cdots \\ a_{m1} & a_{m2} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for an integer $m \geq 2$. Then Theorem 2.4 is refined in the following.

Theorem 3.1 *A linear equation system (LEq2) is non-solvable if and only if one of the following conditions hold:*

(1) *there are integers $1 \leq i, j \leq m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$;*
(2) *there are integers $1 \leq i, j, k \leq m$ such that*

$$\frac{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{k1} & a_{k2} \end{vmatrix}} \neq \frac{\begin{vmatrix} a_{i1} & b_i \\ a_{j1} & b_j \end{vmatrix}}{\begin{vmatrix} a_{i1} & b_i \\ a_{k1} & b_k \end{vmatrix}}.$$

Proof The condition (1) is nothing but the conclusion in Corollary 2.3, i.e., the ith equation is parallel to the jth equation. Now if there no such parallel equations in $(LEq2)$, let T be the elementary transformation replacing all other jth equations by the jth equation plus $(-a_{j1}/a_{i1})$

times the ith equation for integers $1 \leq j \leq m$. We get a transformation $T(A^+)$ of A^+ following

$$T(A^+) = \begin{bmatrix} 0 & \begin{vmatrix} a_{i1} & a_{i2} \\ a_{11} & a_{12} \end{vmatrix} & \begin{vmatrix} a_{i1} & b_i \\ a_{11} & b_1 \end{vmatrix} \\ \cdots & \cdots & \cdots \\ 0 & \begin{vmatrix} a_{i1} & a_{i2} \\ a_{s1} & a_{s2} \end{vmatrix} & \begin{vmatrix} a_{i1} & b_i \\ a_{s1} & b_s \end{vmatrix} \\ a_{i1} & a_{i2} & b_i \\ 0 & \begin{vmatrix} a_{i1} & a_{i2} \\ a_{t1} & a_{t2} \end{vmatrix} & \begin{vmatrix} a_{i1} & b_i \\ a_{t1} & b_t \end{vmatrix} \\ \cdots & \cdots & \cdots \\ 0 & \begin{vmatrix} a_{i1} & a_{i2} \\ a_{m1} & a_{m2} \end{vmatrix} & \begin{vmatrix} a_{i1} & b_i \\ a_{m1} & b_m \end{vmatrix} \end{bmatrix},$$

where $s = i - 1,\ t = i + 1$. Applying Corollary 2.3 again, we know that there are integers $1 \leq i, j, k \leq m$ such that

$$\frac{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix}}{\begin{vmatrix} a_{i1} & a_{i2} \\ a_{k1} & a_{k2} \end{vmatrix}} \neq \frac{\begin{vmatrix} a_{i1} & b_i \\ a_{j1} & b_j \end{vmatrix}}{\begin{vmatrix} a_{i1} & b_i \\ a_{k1} & b_k \end{vmatrix}}.$$

if the linear equation system ($LEQ2$) is non-solvable. □

Notice that a linear equation $ax_1 + bx_2 = c$ with $a \neq 0$ or $b \neq 0$ is a straight line on $\mathbb{R}^2$. We get the following result.

Theorem 3.2 *A liner equation system (LEq2) is non-solvable if and only if one of conditions following hold:*

(1) *there are integers* $1 \leq i, j \leq m$ *such that* $a_{i1}/a_{j1} = a_{i2}/a_{j2} \neq b_i/b_j$*;*

(2) *let* $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ *and*

$$x_1^0 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad x_2^0 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

Then there is an integer $i,\ 1 \leq i \leq m$ *such that*

$$a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) \neq 0.$$

Proof If the linear equation system $(LEq2)$ has a solution (x_1^0, x_2^0), then

$$x_1^0 = \frac{\begin{vmatrix} b_1 & a_{21} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad x_2^0 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

and $a_{i1}x_1^0 + a_{i2}x_2^0 = b_i$, i.e., $a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) = 0$ for any integers $1 \le i \le m$. Thus, if the linear equation system $(LEq2)$ is non-solvable, there must be integers $1 \le i, j \le m$ such that $a_{i1}/a_{j1} = a_{i2}/a_{j2} \ne b_i/b_j$, or there is an integer $1 \le i \le m$ such that

$$a_{i1}(x_1 - x_1^0) + a_{i2}(x_2 - x_2^0) \ne 0.$$

This completes the proof. □

For a non-solvable linear equation system $(LEq2)$, there is a naturally induced *intersection-free graph* $I[LEq2]$ by $(LEq2)$ on the plane $\mathbb{R}^2$ defined following:

$V(I[LEq2]) = \{(x_1^{ij}, x_2^{ij}) | a_{i1}x_1^{ij} + a_{i2}x_2^{ij} = b_i,\ a_{j1}x_1^{ij} + a_{j2}x_2^{ij} = b_j, 1 \le i, j \le m\}$.

$E(I[LEq2]) = \{(v_{ij}, v_{il}) |$the segament between points (x_1^{ij}, x_2^{ij}) and (x_1^{il}, x_2^{il}) in $\mathbb{R}^2\}$. (where $v_{ij} = (x_1^{ij}, x_2^{ij})$ for $1 \le i, j \le m$).

Such an intersection-free graph is clearly a planar graph with each edge a straight segment since all intersection of edges appear at vertices. For example, let the linear equation system be $(LEq2)$ with

$$A^+ = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}.$$

Then its intersection-free graph $I[LEq2]$ is shown in Fig.2.

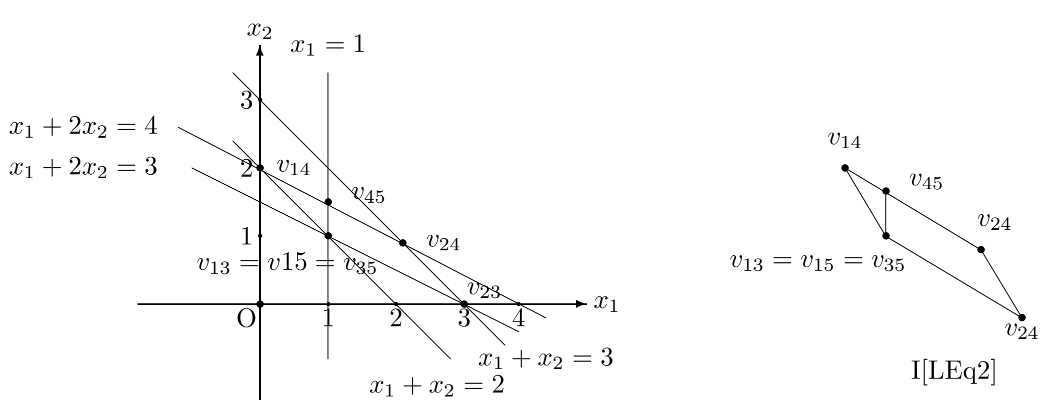

Fig.2

Let H be a planar graph with each edge a straight segment on $\mathbb{R}^2$. Its *c-line graph* $L_C(H)$ is defined by

$V(L_C(H)) = \{\text{straight lines } L = e_1e_2\cdots e_l, s \geq 1 \text{ in } H\}$;

$E(L_C(H)) = \{(L_1, L_2)|$ if e_i^1 and e_j^2 are adjacent in H for $L_1 = e_1^1e_2^1\cdots e_l^1,\ L_2 = e_1^2e_2^2\cdots e_s^2,\ l, s \geq 1\}$.

The following result characterizes the combinatorial structure of non-solvable linear equation systems with two variables by intersection-free graphs $I[LEq2]$.

Theorem 3.3 *A linear equation system* $(LEq2)$ *is non-solvable if and only if*

$$G[LEq2] \simeq L_C(H)),$$

where H *is a planar graph of order* $|H| \geq 2$ *on* $\mathbb{R}^2$ *with each edge a straight segment*

Proof Notice that there is naturally a one to one mapping $\phi : V(G[LEq2]) \rightarrow V(L_C(I[LEq2]))$ determined by $\phi(S_i^0) = S_i^1$ for integers $1 \leq i \leq m$, where S_i^0 and S_i^1 denote respectively the solutions of equation $a_{i1}x_1 + a_{i2}x_2 = b_i$ on the plane $\mathbb{R}^2$ or the union of points between (x_1^{ij}, x_2^{ij}) and (x_1^{il}, x_2^{il}) with

$$\begin{cases} a_{i1}x_1^{ij} + a_{i2}x_2^{ij} = b_i \\ a_{j1}x_1^{ij} + a_{j2}x_2^{ij} = b_j \end{cases}$$

and

$$\begin{cases} a_{i1}x_1^{il} + a_{i2}x_2^{il} = b_i \\ a_{l1}x_1^{il} + a_{l2}x_2^{il} = b_l \end{cases}$$

for integers $1 \leq i, j, l \leq m$. Now if $(S_i^0, S_j^0) \in E(G[LEq2])$, then $S_i^0 \bigcap S_j^0 \neq \emptyset$. Whence,

$$S_i^1 \bigcap S_j^1 = \phi(S_i^0) \bigcap \phi(S_j^0) = \phi(S_i^0 \bigcap S_j^0) \neq \emptyset$$

by definition. Thus $(S_i^1, S_j^1) \in L_C(I(LEq2))$. By definition, ϕ is an isomorphism between $G[LEq2]$ and $L_C(I[LEq2])$, a line graph of planar graph $I[LEq2]$ with each edge a straight segment.

Conversely, let H be a planar graph with each edge a straight segment on the plane $\mathbb{R}^2$. Not loss of generality, we assume that edges $e_{1,2}, \cdots, e_l \in E(H)$ is on a straight line L with equation $a_{L1}x_1 + a_{L2}x_2 = b_L$. Denote all straight lines in H by $\mathscr{C}$. Then H is the intersection-free graph of linear equation system

$$a_{L1}x_1 + a_{L2}x_2 = b_L, \quad L \in \mathscr{C}. \qquad (LEq2^*)$$

Thus,

$$G[LEq2^*] \simeq H.$$

This completes the proof. □

Similarly, we can also consider the liner equation system $(LEq2)$ with condition on x_1 or x_2 such as

$$AX = (b_1, b_2, \cdots, b_m)^T \qquad (L^-Eq2)$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \cdots & \cdots \\ a_{m1} & a_{m2} \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and $x_1 \geq x^0$ for a real number x^0 and an integer $m \geq 2$. In geometry, each of there equation is a ray on the plane $\mathbb{R}^2$, seeing also references [5]-[6]. Then the following conclusion can be obtained like with Theorems 3.2 and 3.3.

Theorem 3.4 *A linear equation system* (L^-Eq2) *is non-solvable if and only if*

$$G[LEq2] \simeq L_C(H)),$$

where H *is a planar graph of order* $|H| \geq 2$ *on* $\mathbb{R}^2$ *with each edge a straight segment.*

§4. Linear Equation Systems with 3 Variables

Let

$$AX = (b_1, b_2, \cdots, b_m)^T \qquad (LEq3)$$

be a linear equation system in 3 variables with

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

for an integer $m \geq 3$. Then Theorem 2.4 is refined in the following.

Theorem 4.1 *A linear equation system (LEq3) is non-solvable if and only if one of the following conditions hold:*

(1) *there are integers* $1 \leq i, j \leq m$ *such that* $a_{i1}/a_{j1} = a_{i2}/a_{j2} = a_{i3}/a_{j3} \neq b_i/b_j$;

(2) *if* (a_{i1}, a_{i2}, a_{i3}) *and* (a_{j1}, a_{j2}, a_{j3}) *are independent, then there are numbers* λ, μ *and an integer* l, $1 \leq l \leq m$ *such that*

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3})$$

but $b_l \neq \lambda b_i + \mu b_j$;

(3) *if* (a_{i1}, a_{i2}, a_{i3}), (a_{j1}, a_{j2}, a_{j3}) *and* (a_{k1}, a_{k2}, a_{k3}) *are independent, then there are numbers* λ, μ, ν *and an integer* l, $1 \leq l \leq m$ *such that*

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3}) + \nu(a_{k1}, a_{k2}, a_{k3})$$

but $b_l \neq \lambda b_i + \mu b_j + \nu b_k$.

Proof By Theorem 2.1, the linear equation system $(LEq3)$ is non-solvable if and only if $1 \leq \text{rank}A \neq \text{rank}A^+ \leq 4$. Thus the non-solvable possibilities of $(LEq3)$ are respectively $\text{rank}A = 1,\ 2 \leq \text{rank}A^+ \leq 4$, $\text{rank}A = 2,\ 3 \leq \text{rank}A^+ \leq 4$ and $\text{rank}A = 3,\ \text{rank}A^+ = 4$. We discuss each of these cases following.

Case 1 $\text{rank}A = 1$ but $2 \leq \text{rank}A^+ \leq 4$

In this case, all row vectors in A are dependent. Thus there exists a number λ such that $\lambda = a_{i1}/a_{j1} = a_{i2}/a_{j2} = a_{i3}/a_{j3}$ but $\lambda \neq b_i/b_j$.

Case 2 $\text{rank}A = 2,\ 3 \leq \text{rank}A^+ \leq 4$

In this case, there are two independent row vectors. Without loss of generality, let (a_{i1}, a_{i2}, a_{i3}) and (a_{j1}, a_{j2}, a_{j3}) be such row vectors. Then there must be an integer $l,\ 1 \leq l \leq m$ such that the lth row can not be the linear combination of the ith row and jth row. Whence, there are numbers λ, μ such that

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3})$$

but $b_l \neq \lambda b_i + \mu b_j$.

Case 3 $\text{rank}A = 3,\ \text{rank}A^+ = 4$

In this case, there are three independent row vectors. Without loss of generality, let $(a_{i1}, a_{i2}, a_{i3}),\ (a_{j1}, a_{j2}, a_{j3})$ and (a_{k1}, a_{k2}, a_{k3}) be such row vectors. Then there must be an integer $l,\ 1 \leq l \leq m$ such that the lth row can not be the linear combination of the ith row, jth row and kth row. Thus there are numbers λ, μ, ν such that

$$(a_{l1}, a_{l2}, a_{l3}) = \lambda(a_{i1}, a_{i2}, a_{i3}) + \mu(a_{j1}, a_{j2}, a_{j3}) + \nu(a_{k1}, a_{k2}, a_{k3})$$

but $b_l \neq \lambda b_i + \mu b_j + \nu b_k$. Combining the discussion of Case 1-Case 3, the proof is complete. $\square$

Notice that the linear equation system $(LEq3)$ can be transformed to the following $(LEq3^*)$ by elementary transformation, i.e., each jth row plus $-a_{j3}/a_{i3}$ times the ith row in $(LEq3)$ for an integer $i,\ 1 \leq i \leq m$ with $a_{i3} \neq 0$,

$$A'X = (b'_1, b'_2, \cdots, b'_m)^T \qquad (LEq3^*)$$

with

$$A'^+ = \begin{bmatrix} a'_{11} & a'_{12} & 0 & b'_1 \\ \cdots & \cdots & \cdots & \cdots \\ a'_{(i-1)1} & a'_{(i-1)2} & 0 & b'_{i-1} \\ a_{i1} & a_{i2} & a_{i3} & b_i \\ a'_{(i+1)1} & a'_{(i+1)2} & 0 & b'_{i+1} \\ a'_{m1} & a'_{m2} & 0 & b'_m \end{bmatrix},$$

where $a'_{j1} = a_{j1} - a_{j3}a_{i1}/a_{13},\ a'_{j2} = a_{j2} - a_{j2}a_{i2}/a_{i3}$ and $b'_j = b_j - a_{j3}b_i/a_{i3}$ fro integers $1 \leq j \leq m$. Applying Theorem 3.3, we get the a combinatorial characterizing on non-solvable linear systems $(LEq3)$ following.

Theorem 4.2 *A linear equation system* $(LEq3)$ *is non-solvable if and only if* $G[LEq3] \not\simeq K_m$ *or* $G[LEq3^*] \simeq u + L_C(H)$*, where* H *denotes a planar graph with order* $|H| \geq 2$*, size* $m-1$ *and each edge a straight segment,* $u+G$ *the join of vertex* u *with* G*.*

Proof By Theorem 2.4, the linear equation system $(LEq3)$ is non-solvable if and only if $G[LEq3] \not\simeq K_m$ or the linear equation system $(LEq3^*)$ is non-solvable, which implies that the linear equation subsystem following

$$BX' = (b'_1, \cdots, b'_{i-1}, b'_{i+1} \cdots, b'_m)^T \qquad (LEq2^*)$$

with

$$B = \begin{bmatrix} a'_{11} & a'_{12} \\ \cdots & \cdots \\ a'_{(i-1)1} & a'_{(i-1)2} \\ a'_{(i+1)1} & a'_{(i+1)2} \\ a'_{m1} & a'_{m2} \end{bmatrix} \quad \text{and} \quad X' = (x_1, x_2)^T$$

is non-solvable. Applying Theorem 3.3, we know that the linear equation subsystem $(LEq2^*)$ is non-solvable if and only if $G[LEq2^*] \simeq L_C(H))$, where H is a planar graph H of size $m-1$ with each edge a straight segment. Thus the linear equation system $(LEq3^*)$ is non-solvable if and only if $G[LEq3^*] \simeq u + L_C(H)$. □

§5. Linear Homeomorphisms Equations

A homeomorphism on $\mathbb{R}^n$ is a continuous $1-1$ mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ such that its inverse h^{-1} is also continuous for an integer $n \geq 1$. There are indeed many such homeomorphisms on $\mathbb{R}^n$. For example, the linear transformations T on $\mathbb{R}^n$. A *linear homeomorphisms equation system* is such an equation system

$$AX = (b_1, b_2, \cdots, b_m)^T \qquad (L^hEq)$$

with $X = (h(x_1), h(x_2), \cdots, h(x_n))^T$, where h is a homeomorphism and

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

for integers $m,\ n \geq 1$. Notice that the linear homeomorphism equation system

$$\begin{cases} a_{i1}h(x_1) + a_{i2}h(x_2) + \cdots a_{in}h(x_n) = b_i, \\ a_{j1}h(x_1) + a_{j2}(x_2) + \cdots a_{jn}h(x_n) = b_j \end{cases}$$

is solvable if and only if the linear equation system

$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots a_{jn}x_n = b_j \end{cases}$$

is solvable. Similarly, two linear homeomorphism equations are said *parallel* if they are non-solvable. Applying Theorems 2.6, 3.3, 4.2, we know the following result for linear homeomorphism equation systems (L^hEq).

Theorem 5.1 *Let (L^hEq) be a linear homeomorphism equation system for integers $m, n \geq 1$. Then*

(1) $G[LEq] \simeq K_{n_1,n_2,\cdots,n_s}$ with $n_1 + n + 2 + \cdots + n_s = m$, where $\mathscr{C}_i^h$ is the parallel family with $n_i = |\mathscr{C}_i^h|$ for integers $1 \leq i \leq s$ in (L^hEq) and (L^hEq) is non-solvable if $s \geq 2$;

(2) If $n = 2$, (L^hEq) is non-solvable if and only if $G[L^hEq] \simeq L_C(H))$, where H is a planar graph of order $|H| \geq 2$ on $\mathbb{R}^2$ with each edge a homeomorphism of straight segment, and if $n = 3$, (L^hEq) is non-solvable if and only if $G[L^hEq] \not\simeq K_m$ or $G[LEq3^*] \simeq u + L_C(H)$, where H denotes a planar graph with order $|H| \geq 2$, size $m-1$ and each edge a homeomorphism of straight segment.

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