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On a note of the Smarandache power function [1]

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Abstract For any positive integer n, the Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that $n|m^m$, where m and n have the same prime divisors. The main purpose of this paper is to study the distribution properties of the $k-th$ power of $SP(n)$ by analytic methods, obtain three asymptotic formulas of $\sum\limits_{n\leq x}(SP(n))^k$, $\sum\limits_{n\leq x}\varphi((SP(n))^k)$ and $\sum\limits_{n\leq x}d(SP(n))^k$ $(k>1)$, and supplement the relate conclusions in some references.

Keywords Smarandache power function, the $k-th$ power, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, we define the Smarandache power function $SP(n)$ as the smallest positive integer m such that $n|m^m$, where n and m have the same prime divisors. That is,

$$SP(n)=\min\left\{m: n|m^m, m\in\mathbb{N}^+, \prod_{p|m}p=\prod_{p|n}p\right\}.$$

If n runs through all natural numbers, then we can get the Smarandache power function sequence $SP(n)$: $1,2,3,2,5,6,7,4,3,10,11,6,13,14,15,4,17,6,19,10,\cdots$, Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, denotes the factorization of n into prime powers. If $\alpha_i<p_i$, for all α_i $(i=1,2,\cdots,r)$, then we have $SP(n)=U(n)$, where $U(n)=\prod\limits_{p|n}p$, $\prod\limits_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function.

In reference [1], Professor F. Smarandache asked us to study the properties of the sequence $SP(n)$. He has done the preliminary research about this question literature [2] $-$ [4], has obtained some important conclusions. And literature [2] has studied an average value, obtained the asymptotic formula:

$$\sum_{n\leq x}SP(n)=\frac{1}{2}x^2\prod_p\left(1-\frac{1}{p(1+p)}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

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Literature [3] has studied the infinite sequence astringency, has given the identical equation:

$$\sum_{n=1}^{\infty}\frac{(-1)^{\mu(n)}}{(SP(n^k))^s}=\begin{cases}\frac{2^s+1}{(2^s-1)\zeta(s)}, & k=1,2;\\ \frac{2^s+1}{(2^s-1)\zeta(s)}-\frac{2^s-1}{4^s}, & k=3;\\ \frac{2^s+1}{(2^s-1)\zeta(s)}-\frac{2^s-1}{4^s}+\frac{3^s-1}{9^s}, & k=4,5.\end{cases}$$

And literature [4] has studied the equation $SP(n^k)=\phi(n), k=1,2,3$ solubility ($\phi(n)$ is the Euler function), and has given all positive integer solution. Namely the equation $SP(n)=\phi(n)$ only has 4 positive integer solutions $n=1,4,8,18$; Equation $SP(n^3)=\phi(n)$ to have and only has 3 positive integer solutions $n=1,16,18$. In this paper, we shall use the analysis method to study the distribution properties of the $k-th$ power of $SP(n)$, gave $\sum\limits_{n\leq x}(SP(n))^k$, $\sum\limits_{n\leq x}\varphi((SP(n))^k)$ and $\sum\limits_{n\leq x}d(SP(n))^k$ $(k>1)$, some interesting asymptotic formula, has promoted the literature [2] conclusion.

Specifically as follows:

Theorem 1.1. For any random real number $x\geq 3$ and given real number k $(k>0)$, we have the asymptotic formula:

$$\sum_{n\leq x}(SP(n))^k=\frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1-\frac{1}{p^k(p+1)}\right)+O(x^{k+\frac{1}{2}+\varepsilon});$$

$$\sum_{n\leq x}\frac{(SP(n))^k}{n}=\frac{\zeta(k+1)}{k\zeta(2)}x^{k}\prod_p\left(1-\frac{1}{p^k(p+1)}\right)+O(x^{k+\frac{1}{2}+\varepsilon}),$$

where $\zeta(k)$ is the Riemann zeta-function, ε denotes any fixed positive number, and $\prod\limits_p$ denotes the product over all primes.

Corollary 1.1. For any random real number $x\geq 3$ and given real number $k'>0$ we have the asymptotic formula:

$$\sum_{n\leq x}(SP(n))^{\frac{1}{k'}}=\frac{6k'\zeta(\frac{1+k'}{k'})}{(k'+1)\pi^2}x^{\frac{1+k'}{k'}}\prod_p\left(1-\frac{1}{(1+p)p^{\frac{1}{k'}}}\right)+O\left(x^{\frac{k'+2}{2k'}+\varepsilon}\right).$$

Specifically, we have

$$\sum_{n\leq x}(SP(n))^{\frac{1}{2}}=\frac{4\zeta(\frac{3}{2})}{\pi^2}x^{\frac{3}{2}}\prod_p\left(1-\frac{1}{(1+p)p^{\frac{1}{2}}}\right)+O(x^{1+\varepsilon});$$

$$\sum_{n\leq x}(SP(n))^{\frac{1}{3}}=\frac{9\zeta(\frac{4}{3})}{2\pi^2}x^{\frac{4}{3}}\prod_p\left(1-\frac{1}{(1+p)p^{\frac{1}{3}}}\right)+O(x^{\frac{5}{6}+\varepsilon}).$$

Corollary 1.2. For any random real number $x\geq 3$, and $k=1,2,3$. We have the asymptotic formula:

$$\sum_{n\leq x}(SP(n))=\frac{1}{2}x^2\prod_p\left(1-\frac{1}{p(1+p)}\right)+O(x^{\frac{3}{2}+\varepsilon});$$

$$\sum_{n\leq x}(SP(n))^2=\frac{6\zeta(3)}{3\pi^2}x^3\prod_p\left(1-\frac{1}{p^2(1+p)}\right)+O(x^{\frac{5}{2}+\varepsilon});$$

$$\sum_{n\leq x}(SP(n))^3=\frac{\pi^2}{60}x^4\prod_p\left(1-\frac{1}{p^3(1+p)}\right)+O(x^{\frac{7}{2}+\varepsilon}).$$

Theorem 1.2. For any random real number $x\geq 3$, we have the asymptotic formula:

$$\sum_{n\leq x}\varphi\big((SP(n))^k\big)=\frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1-\frac{1}{(1+p)p^k}\right)+O(x^{k+\frac{1}{2}+\varepsilon}),$$

where $\varphi(n)$ is the Euler function

Theorem 1.3. For any random real number $x\geq 3$, we have the asymptotic formula:

$$\sum_{n\leq x}d\big((SP(n))^k\big)=B_0x\ln^k x+B_1x\ln^{k-1}x+B_2x\ln^{k-2}x+\cdots+B_{k-1}x\ln x+B_kx+O(x^{\frac{1}{2}+\varepsilon}).$$

where $d(n)$ is the Dirichlet divisor function and $B_0,B_1,B_2,\cdots,B_{k-1},\ B_k$ is computable constant.

§2. Lemmas and proofs

Suppose $s=\sigma+it$ and let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k},U(n)=\prod\limits_{p|n}p$. Before the proofs of the theorem, the following Lemmas will be useful.

Lemma 2.1. For any random real number $x\geq 3$ and given real number $k\geq 1$, we have the asymptotic formula:

$$\sum_{n\leq x}(U(n))^k=\frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1-\frac{1}{(1+p)p^k}\right)+O(x^{k+\frac{1}{2}+\varepsilon}).$$

Proof. Let Dirichlet's series

$$A(s)=\sum_{n=1}^{\infty}\frac{(U(n))^k}{n^s},$$

for any real number $s>1$, it is clear that $A(s)$ is absolutely convergent. Because $U(n)$ is the multiplicative function, if $\sigma>k+1$, so from the Euler's product formula [5] we have

$$\begin{aligned}A(s)&=\sum_{n=1}^{\infty}\frac{(U(n))^k}{n^s}\\&=\prod_p\left(\sum_{m=0}^{\infty}\frac{(U(p^m))^k}{p^{ms}}\right)\\&=\prod_p\left(1+\frac{p^k}{p^s}+\frac{p^k}{p^{2s}}+\cdots\right)\\&=\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}\prod_p\left(1-\frac{1}{p^k(1+p^{s-k})}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Letting $R(k)=\prod\limits_p\left(1-\frac{1}{p^k(1+p^{s-k})}\right)$. If $\sigma>k+1,\ |U(n)|\leq n,\ |\sum\limits_{n=1}^{\infty}\frac{(U(n))^k}{n^\sigma}|<\zeta(\sigma-k)$.

Therefore by Perron's formula [5] with $a(n) = (U(n))^k$, $s_0 = 0$, $b = k + \frac{3}{2}$, $T = x^{k+\frac{1}{2}}$, $H(x) = x$, $B(\sigma) = \zeta(\sigma - k)$, then we have

$$\sum_{n\le x}(U(n))^k = \frac{1}{2\pi i}\int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}+iT}\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s + O(x^{k+\frac{1}{2}+\varepsilon}),$$

where $h(k) = \prod_p \left(1 - \frac{1}{p^k(1+p)}\right)$.

To estimate the main term

$$\frac{1}{2\pi i}\int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}+iT}\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s,$$

we move the integral line from $s = k + \frac{3}{2} \pm iT$ to $k + \frac{1}{2} \pm iT$, then the function

$$\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}$$

have a first-order pole point at $s = k + 1$ with residue

$$\begin{aligned} L(x) &= \operatorname*{Res}_{s=k+1}\left(\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\right) \\ &= \lim_{s\to k+1}\left((s-k-1)\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\right) \\ &= \frac{\zeta(k+1)}{(k+1)\zeta(s)}x^{k+1}h(k). \end{aligned}$$

Taking $T = x^{k+\frac{1}{2}}$, we can easily get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{k+\frac{1}{2}+iT}^{k+\frac{3}{2}+iT} + \int_{k+\frac{1}{2}-iT}^{k+\frac{3}{2}+iT}\right)\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s\right| \ll \frac{x^{2k+1}}{T} = x^{k+\frac{1}{2}},$$

$$\left|\frac{1}{2\pi i}\int_{k+\frac{1}{2}-iT}^{k+\frac{1}{2}+iT}\frac{\zeta(s)\zeta(s-k)}{\zeta(2s-2k)}h(s)\frac{x^s}{s}\mathrm{d}s\right| \ll x^{k+\frac{1}{2}+\varepsilon}.$$

We may immediately obtain the asymptotic formula

$$\sum_{n\le x}(U(n))^k = \frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p\left(1 - \frac{1}{(1+p)p^k}\right) + O(x^{k+\frac{1}{2}+\varepsilon}),$$

this completes the proof of the Lemma 2.1.

Lemma 2.2. For any random real number $x \ge 3$ and given real number $k \ge 1$, and positive integer α, then we have

$$\sum_{\substack{p^\alpha\le x\\ \alpha>p}}(\alpha p)^k \ll \ln^{2k+2} x.$$

Proof. Because $\alpha > p$, so $p^p < p^\alpha \le x$, then

$$p < \frac{\ln x}{\ln p} < \ln x,\ \ \alpha \le \frac{\ln x}{\ln p},$$

also, $\sum_{n\le x} n^k = \frac{x^{k+1}}{k+1} + O(x^k)$. Thus,

$$\sum_{\substack{p^\alpha\le x\\ \alpha>p}} (\alpha p)^k = \sum_{p\le \ln x} p^k \sum_{\alpha\le \frac{\ln x}{\ln p}} \alpha^k \ll \ln^{k+1} x \sum_{p\le \ln x} \frac{p^k}{\ln^{k+1} p} \ll \ln^{k+1} x \sum_{p\le \ln x} p^k.$$

Considering $\pi(x) = \sum_{p\le x} 1$, by virtue of [5], $\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right)$. we can get from the Able

$$\sum_{p\le x} p^k = \pi(x)x^k - k\int_2^x \pi(t)t^{k-1}\mathrm{d}t.$$

Therefore

$$\sum_{p\le \ln x} p^k = \frac{\ln^k x}{(k+1)} + O(\ln^{k-1} x) - k\int_2^{\ln x} \frac{t^k}{\ln t}\mathrm{d}t + O\left(\int_2^{\ln x} \frac{t^k}{\ln^2 t}\mathrm{d}t\right) = \frac{\ln^k x}{k+1} + O(\ln^{k-1} x).$$

Thus

$$\sum_{\substack{p^\alpha\le x\\ \alpha>p}} (\alpha p)^k = \sum_{p\le \ln x} p^k \sum_{\alpha\le \frac{\ln x}{\ln p}} \alpha^k \ll \ln^{k+1} x \sum_{p\le \ln x} \frac{p^k}{\ln^{k+1} p} \ll \ln^{k+1} x \sum_{p\le \ln x} p^k \ll \ln^{2k+2} x.$$

This completes the proof of the Lemma 2.2.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem.

Proof of Theorem 1.1. Let $A = \left\{n|n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, \alpha_i \le p_i, i = 1, 2, \cdots, r\right\}$. When $n \in A : SP(n) = U(n)$; When $n \in \mathbb{N}^+ : SP(n) \ge U(n)$, thus

$$\sum_{n\le x}(SP(n))^k - \sum_{n\le x}(U(n))^k = \sum_{n\le x}\left[(SP(n))^k - (U(n))^k\right] \ll \sum_{\substack{n\le x\\ SP(n)>U(n)}} (SP(n))^k.$$

By the [2] known, there is integer α and prime numbers p, so $SP(n) < \alpha p$, then we can get according to Lemma 2.2

$$\sum_{\substack{n\le x\\ SP(n)>U(n)}} (SP(n))^k < \sum_{\substack{n\le x\\ SP(n)>U(n)}} (\alpha p)^k \ll \sum_{n\le x}\sum_{\substack{p^\alpha<x\\ \alpha>p}} \ll x\ln^{2k+2} x.$$

Therefore

$$\sum_{n\le x}(SP(n))^k - \sum_{n\le x}(U(n))^k \ll x\ln^{2k+2} x.$$

From the Lemma 2.1 we have

$$\begin{aligned}\sum_{n\le x}(SP(n))^k &= \frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p(1-\frac{1}{p^k(1+p)}) + O(x^{k+\frac{1}{2}+\varepsilon}) + O(x\ln^{2k+1} x)\\ &= \frac{\zeta(k+1)}{(k+1)\zeta(2)}x^{k+1}\prod_p(1-\frac{1}{p^k(1+p)}) + O(x^{k+\frac{1}{2}+\varepsilon}).\end{aligned}$$

This proves Theorem 1.1.

Proof of Corollary. According to Theorem 1.1, taking $k = \frac{1}{k'}$ the Corollary 1.1 can be obtained. Take $k = 1, 2, 3$, and $\zeta(2) = \dfrac{\pi^2}{6}, \zeta(4) = \dfrac{\pi^4}{90}$, we can achieve Corollary 1.2. Obviously so is theorem [2].

Using the similar method to complete the proofs of Theorem 1.2 and Theorem 1.3.

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