Scientia Magna
Vol. 3 (2007), No. 4, 104-104

A note on primes of the form $a^2 + 1$

Juan López González

9, December 2007
Madrid, Spain

Abstract In this note I prove using an algebraic identity and Wilson's Theorem that if a^2+1 is an odd prime, thus this prime must has the form $4k^2+1$, then $5 \nmid 2k-3$.

Keywords Pseudo Smarandache function, mean value, asymptotic formula.

If $n = a^2 + 1$ is prime and $n \neq 2$, then n is odd, thus a^2 is even and n must has the form $4k^2 + 1$, where $k \geq 1$ is an integer. The integers $4k^2 + 1$ can be written as

$$4k^2 + 1 = (2k-3)^2 + 3(4k-3) + 1. \tag{1}$$

If $2k - 3 = -1$ then $k = 1$, and $5 \nmid -1$. If $2k - 3 = 1$ then $k = 2$, 17 is a prime with $(2 \cdot 2 - 3, 5) = 1$. If $2k - 3 > 1$ then

$$\begin{aligned} 4k^2 + 1 &\equiv 0 + 3(2k) + 1(\bmod 2k-3) \\ &\equiv 3(2k-3) + 9 + 1(\bmod 2k-3) \\ &\equiv 10 \ (\bmod 2k-3). \end{aligned}$$

By Wilson's Theorem

$$(4k^2)! \equiv -1(\bmod 4k^2+1). \tag{2}$$

Thus exists an integer c such that $(4k^2)! + 1 = c \cdot (4k^2 + 1)$, since $4k^2 > 2k - 3$ for all k, then $2k - 3 | (4k^2)!$, thus

$$0 + 1 \equiv c \cdot 10(\bmod 2k-3). \tag{3}$$

Then there are integers s and t, such that

$$10s + (2k-3)t = 1, \tag{4}$$

thus $(5, 2k-3) = 1$, by contradiction if $5 | 2k-3$, then $0 + 0 \equiv 1(\bmod 5)$. Thus I've proved the following

Proposition. If $a^2 + 1$ is an odd prime different of 5, then $(a - 3, 5) = 1$.

References

[1] Richard K. Guy, Unsolved Problems in Number Theory, Springer, Second Edition, **I**(1994).

Scientia Magna
Vol. 3 (2007), No. 4, 105-110

The natural partial order on U-semiabundant semigroups[1]

Dehua Wang†, Xueming Ren† and X. L. Ding‡

† Department of Mathematics, Xi'an University of Architecture and Technology
Xi'an, 710055, P.R. China
Email: xmren@xauat.edu.cn

‡ Department of Mathematics, Inner Mongolia University of Science and Technology
Baotou 014100, P.R. China

Abstract The natural partial order on an U-semiabundant semigroup is introduced in this paper and some properties of U-semiabundant semigroups are investigated by the natural partial order. In addition, we also discuss a special class of U-semiabundant semigroups in which the natural partial order is compatible with the multiplication.

Keywords U-semiabundant semigroups, natural partial orders.

§1. Introduction

In generalizing regular semigroups, a generalized Green relation $\widetilde{\mathcal{L}}^U$ was introduced by M. V. Lawson [4] on a semigroup S as follows:

Let E be the set of all idempotents of S and U be a subset of E. For any $a, b \in S$, define

$$(a,b) \in \widetilde{\mathcal{L}}^U \quad \text{if and only if} \quad (\forall\ e \in U) \quad (\ ae = a \Leftrightarrow be = b\ );$$

$$(a,b) \in \widetilde{\mathcal{R}}^U \quad \text{if and only if} \quad (\forall\ e \in U) \quad (\ ea = a \Leftrightarrow eb = b\ ).$$

It is clear that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}^U$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}^U$.

It is easy to verify that if S is an abundant semigroup and $U = E(S)$ then $\mathcal{L}^* = \widetilde{\mathcal{L}}^U$, $\mathcal{R}^* = \widetilde{\mathcal{R}}^U$; if S is a regular semigroup and $U = E(S)$ then $\mathcal{L} = \widetilde{\mathcal{L}}^U$, $\mathcal{R} = \widetilde{\mathcal{R}}^U$.

Recall that a semigroup S is called U-semiabundant if each $\widetilde{\mathcal{L}}^U$-class and each $\widetilde{\mathcal{R}}^U$-class contains an element from U.

It is clear that regular semigroups and abundant semigroups are all U-semiabundant semigroups.

The natural partial order on a regular semigroup was first studied by Nambooripad [7] in 1980. Later on, M. V. Lawson [1] in 1987 first introduced the natural partial order on an abundant semigroup. The partial orders on various kinds of semigroups have been investigated by many authors, for example, H. Mitsch [5], Sussman [6], Abian [8] and Burgess[9].

[1]This research is supported by National Natural Science Foundation of China (Grant No:10671151)

In this paper, we will introduce the natural partial order on U-semiabundant semigroups and describe the properties of such semigroups by using the natural partial order.

We first cite some basic notions which will be used in this paper. Suppose that $e,\ f$ are elements of $E(S)$. The preorders ω^r and ω^l are defined as follows:

$$e\omega^r f \Leftrightarrow fe = e \quad \text{and} \quad e\omega^l f \Leftrightarrow ef = e.$$

In addition, $\omega = \omega^r \cap \omega^l$, the usual ordering on $E(S)$.

We use $\mathcal{D}_E$ to denote the relation $(\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$. Assume (S, U) is an U-semiabundant semigroup.

It will be said that U is closed under basic products if $e, f \in U$ and $(e, f) \in \mathcal{D}_E$ then $ef \in U$.

For terminologies and notations not given in this paper, the reader is referred to Howie [3].

§2. The natural partial order

Let $S(U)$ be an U-semiabundant semigroup and $a \in S$. The $\widetilde{\mathcal{L}}^U(\widetilde{\mathcal{R}}^U)$- class containing the element a will be denoted by $\widetilde{L}^U_a(\widetilde{R}^U_a)$ respectively.

We will denote an element of $\widetilde{L}^U_a \cap U$ by a^* and an element of $\widetilde{R}^U_a \cap U$ by a^+.

Recall in [4] that a right ideal I of a semigroup S is said to be an U-admissible right ideal if for every $a \in I$ we have $\widetilde{R}^U_a \subseteq I$.

For $a \in S$, we define the principal U-admissible right ideal containing a, denoted by $\widetilde{R}^U(a)$, to be the intersection of all U-admissible right ideals containing a. Similarly, we may give the definitions of an U-admissible left ideal and the principal U-admissible left ideal.

Let S be a semigroup and $x,\ y \in S$. We say that $\widetilde{R}^U_x \leqslant \widetilde{R}^U_y$ if $\widetilde{R}^U(x) \subseteq \widetilde{R}^U(y)$. A partial order on the $\widetilde{\mathcal{L}}^U$-classes can be defined in the usual left-right dual way.

Lemma 2.1. $\widetilde{R}^U_{ax} \leqslant \widetilde{R}^U_a$, for any elements a and x of S.

Proof. Clearly, the product ax lies in aS, which is the smallest right ideal containing a. Since $\widetilde{R}^U(a)$ is a right ideal containing a, we have $aS \subseteq \widetilde{R}^U(a)$. Thus $ax \in \widetilde{R}^U(a)$.

It follows immediately that

$$\widetilde{R}^U(ax) \subseteq \widetilde{R}^U(a).$$

Lemma 2.2. Let $U \subseteq E(S)$ and $e,\ f \in U$. Then $\widetilde{R}^U_e \leqslant \widetilde{R}^U_f$ if and only if $R_e \leqslant R_f$.

Proof. Suppose first that $\widetilde{R}^U_e \leqslant \widetilde{R}^U_f$. Then we immediately have $\widetilde{R}^U(e) \subseteq \widetilde{R}^U(f)$. We claim that eS is an U-admissible right ideal.

In fact, for each $a \in eS$, $a = ea$ and so, for any $b \in \widetilde{R}^U_a$, we have $b = eb \in eS$. But $\widetilde{R}^U(e)$ is a right ideal and $e \in U$, so that $eS \subseteq \widetilde{R}^U(e)$.

Since eS is an U-admissible right ideal, we have that $\widetilde{R}^U(e) = eS$. Similarly $\widetilde{R}^U(f) = fS$. It follows that $eS \subseteq fS$, that is, $R(e) \subseteq R(f)$. Hence $R_e \leqslant R_f$.

Conversely, suppose that $R_e \leqslant R_f$. Then $eS \subseteq fS$ and so $e = fx$ for some x in S^1. Thus, by Lemma 2.1, we have $\widetilde{R}^U_e = \widetilde{R}^U_{fx} \leqslant \widetilde{R}^U_f$.

Corollary 2.3. The following statements hold on an U-semiabundant semigroup $S(U)$ for any $e, f \in U$:

(i) $(e,f) \in \widetilde{\mathcal{L}}^U$ if and only if $(e,f) \in \mathcal{L}$;

(ii) $(e,f) \in \widetilde{\mathcal{R}}^U$ if and only if $(e,f) \in \mathcal{R}$.

Theorem 2.4. Let $S(U)$ be an U-semiabundant semigroup such that U is closed under basic products. Define two relations on $S(U)$ as follows:

For any x and y of $S(U)$,

$x \widetilde{\leqslant}_r y$ if and only if $\widetilde{R}_x^U \leqslant \widetilde{R}_y^U$ and there exists an idempotent $x^+ \in \widetilde{R}_x^U \cap U$ such that $x = x^+y$;

$x \widetilde{\leqslant}_l y$ if and only if $\widetilde{L}_x^U \leqslant \widetilde{L}_y^U$ and there exists an idempotent $x^* \in \widetilde{L}_x^U \cap U$ such that $x = yx^*$.

Then $\widetilde{\leqslant}_r$ and $\widetilde{\leqslant}_l$ are respectively two partial orders on $S(U)$ which coincide with ω on U.

Proof. We only need to prove that $\widetilde{\leqslant}_r$ is a partial order on $S(U)$ which coincides with ω on U since the proof of $\widetilde{\leqslant}_l$ is similar.

Reflexivity follows from the fact that $S(U)$ is U-semiabundant. Now suppose that $x \widetilde{\leqslant}_r y$ and $y \widetilde{\leqslant}_r x$. Then $\widetilde{R}_x^U = \widetilde{R}_y^U$ and there exist $x^+ \in \widetilde{R}_x^U \cap U$ and $y^+ \in \widetilde{R}_y^U \cap U$ such that $x = x^+y$ and $y = y^+x$. By Corollary 2.3, we have $x = (x^+y^+)x = y^+x = y$. Next, suppose that $x \widetilde{\leqslant}_r y$ and $y \widetilde{\leqslant}_r z$.

It follows that $\widetilde{R}_x^U \leqslant \widetilde{R}_y^U \leqslant \widetilde{R}_z^U$ and there exist $x^+ \in \widetilde{R}_x^U \cap U$ and $y^+ \in \widetilde{R}_y \cap U$ such that $x = x^+y$ and $y = y^+z$. Thus $x = (x^+y^+)z$ and $\widetilde{R}_{x^+}^U = \widetilde{R}_x^U \leqslant \widetilde{R}_y^U = \widetilde{R}_{y^+}^U$ which gives $R_{x^+} \leqslant R_{y^+}$ by Lemma 2.2.

It follows that $x^+S(U) \subseteq y^+S(U)$ and so $x^+ = y^+x^+$. Since U is closed under basic products and $(x^+, y^+) \in \omega^r \subseteq \mathcal{D}_E$, we deduce that $x^+y^+ \in U$. Clearly, $(x^+, x^+y^+) \in \mathcal{R}$ and so $(x, x^+y^+) \in \widetilde{\mathcal{R}}^U$ by Corollary 2.3. This leads to $x \widetilde{\leqslant}_r z$.

In fact, we have already shown that $\widetilde{\leqslant}_r$ is a partial order on an U-semiabundant semigroup $S(U)$. It is easy to verify that $\widetilde{\leqslant}_r$ coincides with the order ω on U.

Now the natural partial order $\widetilde{\leqslant}$ on an U-semiabundant semigroup $S(U)$ is defined by $\widetilde{\leqslant} = \widetilde{\leqslant}_r \cap \widetilde{\leqslant}_l$. We first give an alternative description of the natural partial order $\widetilde{\leqslant}$ in terms of idempotents.

Theorem 2.5. Let $S(U)$ be an U-semiabundant semigroup such that U is closed under basic products and $x, y \in S(U)$. Then $x \widetilde{\leqslant} y$ if and only if there exist idempotents e and f in U such that $x = ey = yf$.

Proof. We first prove the sufficiency part of Theorem 2.5. Suppose that $x = ey = yf$. From $x = yf$ and Lemma 2.1 we have $\widetilde{R}_x^U \leqslant \widetilde{R}_y^U$. Choosing an idempotent x^+ in $\widetilde{R}_x^U \cap U$, we obtain that $x = x^+x = (x^+e)y$.

Since $ex = x$ and $(x, x^+) \in \widetilde{\mathcal{R}}^U$, we have $ex^+ = x^+$. This implies $(x^+, e) \in \omega^r \subseteq \mathcal{D}_E$. By assumption, $x^+e \in U$. Certainly, $(x^+, x^+e) \in \mathcal{R}$ and so $(x^+, x^+e) \in \widetilde{\mathcal{R}}^U$ by Corollary 2.3. Thus $(x, x^+e) \in \widetilde{\mathcal{R}}^U$. Hence $x \widetilde{\leqslant}_r y$. A similar argument shows that $x \widetilde{\leqslant}_l y$.

The necessity part of Theorem 2.5 is straightforward from Theorem 2.4.

Theorem 2.6. Let $S(U)$ be an U-semiabundant semigroup in which U is closed under basic products and $x, y \in S(U)$. Then $x \widetilde{\leqslant}_r y$ if and only if for each idempotent $y^+ \in \widetilde{R}_y^U \cap U$ there exists an idempotent $x^+ \in \widetilde{R}_x^U \cap U$ such that $x^+ \omega y^+$ and $x = x^+y$. The dual result holds for $\widetilde{\leqslant}_l$.

Proof. Suppose that $x\widetilde{\leqslant}_r y$. Then $\widetilde{R}^U_x \leqslant \widetilde{R}^U_y$ and $x = ey$ for some idempotent $e \in \widetilde{R}^U_x \cap U$ by Theorem 2.4.

Let f be an idempotent in $\widetilde{R}^U_y \cap U$. Then $\widetilde{R}^U_e = \widetilde{R}^U_x \leqslant \widetilde{R}^U_y = \widetilde{R}^U_f$ and so, by Lemma 2.2, $R_e \leqslant R_f$. This leads to $eS(U) \subseteq fS(U)$ and so $e = fe$ giving $ef \in U$ by hypothesis. Clearly, $(e, ef) \in \mathcal{R}$ which gives $ef\widetilde{\mathcal{R}}^U e\widetilde{\mathcal{R}}^U x$ by Corollary 2.3. Hence $ef\omega f$ and $x = ey = (ef)y$, where $ef \in \widetilde{R}^U_x \cap U$.

Conversely, suppose that for each idempotent $y^+ \in \widetilde{R}^U_y \cap U$ there exists an idempotent $x^+ \in \widetilde{R}^U_x \cap U$ such that $x^+\omega y^+$ and $x = x^+y$. Then $x = y^+x^+y$ and so $\widetilde{R}^U_x = \widetilde{R}^U_{y^+x^+y} \leqslant \widetilde{R}^U_{y^+} = \widetilde{R}^U_y$. By Theorem 2.4, $x\widetilde{\leqslant}_r y$. The proof is completed.

§3. Locally V-semiadequate semigroups

In this section we want to find the conditions on an U-semiabundant semigroup $S(U)$ which make that the natural partial order $\widetilde{\leqslant}$ is compatible with multiplication of $S(U)$.

Recall in [2] that an U-semiabundant semigroup $S(U)$ is called reduced if $\omega^r = \omega^l$ on U. A reduced U-semiabundant semigroup $S(U)$ is idempotent connected(IC) if it satisfies the two equations

IC_l: For any $f \in \omega(x^*) \cap U$, $xf = (xf)^+x$;

IC_r: For any $e \in \omega(x^+) \cap U$, $ex = x(ex)^*$.

Lemma 3.1. Let $S(U)$ be a reduced U-semiabundant semigroup then

(i) If IC_l holds then $\widetilde{\leqslant}_l \subseteq \widetilde{\leqslant}_r$;

(ii) If IC_r holds then $\widetilde{\leqslant}_r \subseteq \widetilde{\leqslant}_l$.

Proof. We only need to prove (i) because the proof of (ii) is similar. If $x\widetilde{\leqslant}_l y$ then $x^*\omega y^*$ and $x = yx^*$ by the dual result of Theorem 2.6. Thus, by applying the condition IC_l, we can obtain $x = yx^* = (yx^*)^+y = x^+y$.

Certainly, $y^+(yx^*) = yx^*$ and so $y^+(yx^*)^+ = (yx^*)^+$. Since $S(U)$ is a reduced U-semiabundant semigroup, we can easily see that $x^+ = (yx^*)^+\omega y^+$. It follows from Theorem 2.6 that $x\widetilde{\leqslant}_r y$.

Lemma 3.2. Let $S(U)$ be an U-semiabundant semigroup in which U is closed under basic products and $e \in U$. Then $eS(U)e$ is a V-semiabundant semigroup, where $V = U \cap E(eS(U)e)$.

Proof. Let a be an element of $eS(U)e$ and let f be an element of U with $(f, a) \in \widetilde{\mathcal{L}}^U$. Certainly, $ae = a$ so that $fe = f$, that is, $(e, f) \in (\omega^l)^{-1} \subseteq \mathcal{D}_E$.

Since U is closed under basic products, the element $ef \in V$. Clearly, $(ef, f) \in \mathcal{L}$ so that $(ef, f) \in \widetilde{\mathcal{L}}^U$ by Corollary 2.3. It is easy to verify that $(ef, a) \in \widetilde{\mathcal{L}}^V(eS(U)e)$. This implies that each element of $eS(U)e$ is $\widetilde{\mathcal{L}}^V$-related in $eS(U)e$ to an idempotent belonging to V.

A similar result for $\widetilde{\mathcal{R}}^V$ gives us the required V-semiabundancy.

An U-semiabundant semigroup $S(U)$ is said to be $\widetilde{\mathcal{L}}^U$-unipotent if U forms a right regular band. $S(U)$ is called U-semiadequate if U forms a semilattice.

For any $e \in U$, we call $eS(U)e$ a local submonoid of $S(U)$. We shall say that $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent(locally V-semiadequate) if every local submonoid is $\widetilde{\mathcal{L}}^V$-unipotent($V$-semiadequate).

A subset A of a poset $(X, \widetilde{\leqslant})$ is said to be an order ideal if for each $a \in A$ and for any $x \in X$ with $x \widetilde{\leqslant} a$ then $x \in A$(see [1]). An U-semiabundant semigroup satisfies the congruence condition if $\widetilde{\mathcal{L}}^U$ and $\widetilde{\mathcal{R}}^U$ are right and left congruences on an U-semiabundant semigroup, respectively (see [4]).

Now we arrive at the main result of this section.

Theorem 3.3. Let $S(U)$ be an IC reduced U-semiabundant semigroup, in which U is closed under basic products, satisfying the two conditions:

(C1) For any $e \in U$, $U \cap eS(U)e$ is an order ideal of $E \cap eS(U)e$;

(C2) The congruence condition holds.

Then the natural partial order $\widetilde{\leqslant}$ is right compatible with the multiplication if and only if $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent, where $V = U \cap eS(U)e$.

Proof. Suppose first that the natural partial order $\widetilde{\leqslant}$ is right compatible and $x, y \in V$. Then $x \widetilde{\leqslant} e$ and so $xy \widetilde{\leqslant} ey = y$.

Thus, by Theorem 2.6, there exists $f \in U$ such that $xy = yf = y(yf) = yxy$. It follows that $(xy)(xy) = x(yxy) = x(xy) = xy$ and so that $xy \in E \cap eS(U)e$. According to (C1), $xy \in V$. We have shown that V forms a right regular band. But, by Lemma 3.2, the local submonoid $eS(U)e$ is V-semiabundant. Hence $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent.

Conversely, suppose that $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent, that is, for any $e \in U$, $V = U \cap eS(U)e$ forms a right regular band and $a, b, c \in S(U)$ with $a \widetilde{\leqslant} b$. Then $a \widetilde{\leqslant}_r b$ and so for each idempotent$b^+ \in \widetilde{R}^U_b \cap U$ there exists an idempotent $a^+ \in \widetilde{R}^U_a \cap U$ such that $a^+ \omega b^+$ and $a = a^+ b$.

Since $(bc, (bc)^+) \in \widetilde{\mathcal{R}}^U$ and $b^+(bc) = bc$, we have $b^+(bc)^+ = (bc)^+$. By the hypothesis that U is closed under basic products, $(bc)^+ b^+ \in U \cap b^+ S(U) b^+$. Certainly, $((bc)^+ b^+, (bc)^+) \in \mathcal{R}$ so that $(bc)^+ b^+ \widetilde{\mathcal{R}}^U (bc)^+ \widetilde{\mathcal{R}}^U bc$ by Corollary 2.3.

According to (C2), $(a^+(bc)^+ b^+, ac) = (a^+(bc)^+ b^+, a^+ bc) \in \widetilde{\mathcal{R}}^U$. Since $U \cap b^+ S(U) b^+$ is a right regular band and $a^+ \omega b^+$, we have $a^+(bc)^+ b^+ \in U \cap b^+ S(U) b^+$ and $a^+(bc)^+ b^+ = (bc)^+ b^+ a^+ (bc)^+ b^+$. Again, $ac = a^+ bc = [a^+(bc)^+ b^+] bc$, where $a^+(bc)^+ b^+ \in \widetilde{R}^U_{ac} \cap U$.

Thus

$$\widetilde{R}^U_{ac} = \widetilde{R}^U_{[a^+(bc)^+ b^+] bc} \leqslant \widetilde{R}^U_{a^+(bc)^+ b^+} = \widetilde{R}^U_{(bc)^+ b^+ a^+ (bc)^+ b^+} \leqslant \widetilde{R}^U_{(bc)^+} = \widetilde{R}^U_{bc}.$$

It follows from Theorem 2.4 that $ac \widetilde{\leqslant}_r bc$.

By Lemma 3.1, we also have $\widetilde{\leqslant}_r = \widetilde{\leqslant}_l$. Hence $ac \widetilde{\leqslant} bc$, as required.

Combining Theorem 3.3 with its dual, we may obtain

Corollary 3.4. Let $S(U)$ be an IC reduced U-semiabundant semigroup in which U is closed under basic products. If

(C1) For any $e \in U$, $U \cap eS(U)e$ is an order ideal of $E \cap eS(U)e$,

(C2) $S(U)$ satisfies the congruence condition,

then the natural partial order $\widetilde{\leqslant}$ is compatible with the multiplication if and only if $S(U)$ is locally V-semiadequate, where $V = U \cap eS(U)e$.

References

[1] M. V. Lawson, The natural partial order on an abundant semigroup, Proc. Edinburgh Math. Soc., **30**(1987), 169-186.

[2] M. V. Lawson, Semigroups and ordered categories, Journal of Algebra, **141**(1991), 422-462.

[3] J. M. Howie, An introduction to semigroup theory, Academic Press, 1976.

[4] M. V. Lawson, Rees matrix semigroups, Proc. Edinburgh Math. Soc., **33**(1990), 23-37.

[5] H. Mitsch, A natural partial order for semigroups, Proc. Amer. Math. soc., **97**(1986), 384-388.

[6] I. A. Sussman, A generalisation of Boolean rings, Math. Ann., **136**(1958), 326-338.

[7] K. S. S. Nambooripad, The natural partial order on a regular semigroup, Proc. Edinburgh Math. Soc., **23**(1980), 249-260.

[8] A. Abian, Direct product decomposition of commutative semisimple rings, Proc. Amer. Math. Soc., **24**(1970), 502-507.

[9] W. D. Burgess, Completions of semilattices of cancellative semigroups, Glasgow Math. J., **21**(1980), 29-37.