THE 97-TH PROBLEM OF F.SMARANDACHE *

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Abstract The main purpose of this paper is using the analytic method to study the n-ary sieve sequence, and solved one conjecture about this sequence.

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§1. Introduction and results

In 1991, American-Romanian number theorist Florentin Smarandache introduced hundreds of interesting sequences and arithmetical functions, and presented 105 unsolved arithmetical problems and conjectures about these sequences and functions in book [1]. Already many researchers studied these sequences and functions from this book, and obtained important results. Among these problems, the 97-th unsolved problem is:

Let n be any positive integer with $n \geq 2$, starting to count on the natural numbers set at any step from 1:

— delete every n-th number;

— delete from the remaining ones, every (n^2)-th number;

$\cdots\cdots$;

and so on: delete from the remaining ones, every (n^k)-th number, $k = 1, 2, 3, \cdots$.

For this special sequence, there are two conjectures:

(1) there are an infinity of primes that belong to this sequence;

(2) there are an infinity of number of this sequence which are not prime.

In this paper, we shall use the analytic method to study the n-ary sieve sequence, and solved conjecture (2). That is, we have the following conclusion:

Theorem. For any positive integer $n \geq 2$, the conjecture (2) of n-ary sequence is true.

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§2. Proof of Theorem

In this section, we shall complete the proof of Theorem. For any fixed real number $x \geq 1$ and positive integer k, let $\mathcal{A}_k(x)$ denotes the number of remaining ones after deleting (n^k)-th number from the interval $[1, x]$. In the interval $[1, x]$, for any $n \in [1, x]$, first we delete n-th number from the interval $[1, x]$, then the number of remaining ones is

$$\mathcal{A}_1(x) = [x] - \left[\frac{x}{n}\right],$$

where $[x]$ denotes the greatest integer which is not exceeding x, and $x - 1 \leq [x] \leq x + 1$.

Note that

$$\mathcal{A}_1(x) = [x] - \left[\frac{x}{n}\right] \leq x + 1 - \frac{x}{n} = x\left(1 - \frac{1}{n}\right) + 1, \tag{1}$$

if we delete every (n^2)-th number from the remaining ones, then the number of remaining ones is

$$\mathcal{A}_2(x) = [x] - \left[\frac{x}{n}\right] - \left[\frac{[x] - [\frac{x}{n}]}{n^2}\right].$$

From (1), we have the inequality

$$\begin{aligned} & [x] - \left[\frac{x}{n}\right] - \left[\frac{[x] - [\frac{x}{n}]}{n^2}\right] \\ \leq\ & \left[x\left(1 - \frac{1}{n}\right) + 1\right] - \left[\frac{x\left(1 - \frac{1}{n}\right) + 1}{n^2}\right] \\ \leq\ & x\left(1 - \frac{1}{n}\right) + 2 - \frac{x\left(1 - \frac{1}{n}\right) + 1}{n^2} \\ =\ & x\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right) + \left(2 - \frac{1}{n^2}\right) \\ \leq\ & x\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right) + 2. \end{aligned} \tag{2}$$

$\cdots\cdots$, and so on: if we delete every (n^k)-th number, from the remaining ones, we also have the inequality

$$\mathcal{A}_k(x) = x\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n^2}\right)\cdots\left(1 - \frac{1}{n^k}\right) + k. \tag{3}$$

Similarly, we can also deduce that

$$x\left(1 - \frac{1}{n}\right) - 1 = x - 1 - \frac{x}{n} \leq \mathcal{A}_1(x) = [x] - \left[\frac{x}{n}\right], \tag{4}$$

$$x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)-2\leq \mathcal{A}_2(x)=[x]-\left[\frac{x}{n}\right]-\left[\frac{[x]-\left[\frac{x}{n}\right]}{n^2}\right], \qquad (5)$$

$\cdots\cdots$, and so on:

$$x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)-k\leq \mathcal{A}_k(x). \qquad (6)$$

Combining (5) and (6), we have the asymptotic formula

$$\mathcal{A}_k(x)=x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)+O(k). \qquad (7)$$

Note that $k \ll \ln x$, so we have

$$\mathcal{A}_k(x)=x\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)+O(\ln x). \qquad (8)$$

Let $\pi(x)$ denotes the number of the primes up to x, then we have (see reference [2])

$$\pi(x)=\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right). \qquad (9)$$

Note that $\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{1}{n^k}\right)$ is convergence if $k\to+\infty$, so

$$\mathcal{A}_k(x)-\pi(x)\to+\infty, \qquad \text{if} \quad x\to+\infty.$$

That is, there are an infinity of number of this sequence which are not prime.

This completes the proof of Theorem.

References

[1] F. Smarandache, Only Problems, Not Solutions, Xiquan Publishing House Chicago, 1993.

[2] Tom M. Apostol, Introduction to Analytic Number Theory, New York, Springer-Verlag, 1976.