Scientia Magna
Vol. 3 (2007), No. 3, 1-4

Some faces of Smarandache semigroups' concept in transformation semigroups' approach

F. Ayatollah Zadeh Shirazi† and A. Hosseini‡

† Fatemah Shirazi Ayatollah Zadeh Shirazi, Faculty of Mathematics, Statistics, and Computer Science, College of Science, University of Tehran
Enghelab Ave., Tehran, Iran
e-mail: fatemah@khayam.ut.ac.ir

‡ Arezoo Hosseini, Department of Mathematics, Faculty of Science, Guilan University
Manzarieh Ave., Rasht, Iran

Abstract In the following text, the main aim is to distinguish some relations between Smarandache semigroups and (topological) transformation semigroups areas. We will see that a transformation group is not distal if and only if its enveloping semigroup is a Smarandache semigroup. Moreover we will find a classifying of minimal right ideals of the enveloping semigroup of a transformation semigroup.

Keywords $A-$minimal set, distal, Smarandache semigroup, transformation semigroup.

§1. Preliminaries

By a transformation semigroup (group) (X, S, π) (or simply (X, S)) we mean a compact Hausdorff topological space X, a discrete topological semigroup (group) S with identity e and a continuous map $\pi : X \times S \to X$ ($\pi(x, s) = xs$ ($\forall x \in X, \forall s \in S$)) such that:

1) $\forall x \in X, \quad xe = x$,

2) $\forall x \in X, \quad \forall s, t \in S, \quad x(st) = (xs)t$.

In the transformation semigroup (X, S) we have the following definitions:

1. For each $s \in S$, define the continuous map $\pi^s : X \to X$ by $x\pi^s = xs$ ($\forall x \in X$), then E(X,S)(or simply E(X)) is the closure of $\{\pi^s | \, s \in S\}$ in X^X with pointwise convergence, moreover it is called the enveloping semigroup (or Ellis semigroup) of (X, S). We used to write s instead of π^s ($s \in S$). E(X,S) has a semigroup structure [1, Chapter 3], a nonempty subset K of E(X,S) is called a right ideal if KE(X,S)$\subseteq K$, and it is called a minimal right ideal if none of the right ideals of E(X,S) be a proper subset of K. For each $p\in$E(X), L_p : E(X)$\to$ E(X) with $L_p(q) = pq$ ($q \in$ E(X)) is continuous.

2. Let $a \in X$, A be a nonempty subset of X and K be a closed right ideal of E(X,S)[2, Definition 1]:

1) We say K is an $a-$minimal set if: $aK = a\mathrm{E}(X, S)$, K does not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X, S)$ and $aL = a\mathrm{E}(X, S)$. The set of all $a-$minimal sets is denoted by $\mathrm{M}_{(X,S)}(a)$.

2) We say K is an $A-\overline{\text{minimal}}$ set if: $\forall b \in A\ bK = b\mathrm{E}(X,S)$, K does not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X,S)$ and $bL = b\mathrm{E}(X,S)$ for all $b \in A$. The set of all $A-\overline{\text{minimal}}$ sets is denoted by $\overline{\mathrm{M}}_{(X,S)}(A)$.

3) We say K is an $A-\overline{\overline{\text{minimal}}}$ set if: $AK = A\mathrm{E}(X,S)$, K does not have any proper subset like L, such that L is a closed right ideal of $\mathrm{E}(X,S)$ and $AL = A\mathrm{E}(X,S)$.

The set of all $A-\overline{\overline{\text{minimal}}}$ sets is denoted by $\overline{\overline{\mathrm{M}}}_{(X,S)}(A)$. $\overline{\mathrm{M}}_{(X,S)}(A)$ and $\mathrm{M}_{(X,S)}(a)$ are nonempty.

3. Let $a \in X$, A be a nonempty subset of X [2, Definition 13]:

1) (X,S) is called distal if $\mathrm{E}(X,S)$ is a minimal right ideal.

2) (X,S) is called $a-$distal if $\mathrm{M}_{(X,S)}(a) = \{\mathrm{E}(X,S)\}$.

3) (X,S) is called $A\underline{^{(\overline{\mathrm{M}})}}$distal if $\overline{\mathrm{M}}_{(X,S)}(A) = \{\mathrm{E}(X,S)\}$.

4) (X,S) is called $A\underline{^{(\overline{\overline{\mathrm{M}}})}}$distal if $\overline{\overline{\mathrm{M}}}_{(X,S)}(A) = \{\mathrm{E}(X,S)\}$.

4. Let $a \in X$, A be a nonempty subset of X and C be a nonempty subset of $\mathrm{E}(X,S)$, we introduce the following notations $\mathrm{F}(a,C) = \{p \in C | ap = p\}$, $\mathrm{F}(A,C) = \{p \in C | \forall b \in A\ bp = p\}$, $\overline{\mathrm{F}}(A,C) = \{p \in C |\ Ap = A\}$, $\mathrm{J}(C) = \{p \in C |\ p^2 = p\}$ (the set of all idempotents of C), $\overline{\mathrm{M}}(X,S) = \{\emptyset \neq B \subseteq X |\ \forall K \in \overline{\mathrm{M}}_{(X,S)}(B)\ \mathrm{J}(\mathrm{F}(B,K)) \neq \emptyset\}$, and $\overline{\overline{\mathrm{M}}}(X,S) = \{\emptyset \neq B \subseteq X |\ \overline{\overline{\mathrm{M}}}_{(X,S)}(B) \neq \emptyset \wedge (\forall K \in \overline{\overline{\mathrm{M}}}_{(X,S)}(B)\ \mathrm{J}(\mathrm{F}(B,K)) \neq \emptyset)\}$.

In the following text in the transformation semigroup (X,S) suppose S acts effective on X, i.e., for each $s,\ t \in S$ if $s \neq t$, then there exists $x \in X$ such that $xs \neq xt$. Moreover consider S is a Smarandache semigroup if S has a semigroup structure which does not provides a group structure on S, although S has at least a proper subset with more than one element, which carries a group structure induced by the action considered on S [3, Chapter 4].

§2. Proof of the theorems

Theorem 1. Transformation group (X,G) (with $|G| > 1$) is not distal if and only if $\mathrm{E}(X,G)$ is a Smarandache semigroup.

Proof. If $\mathrm{E}(X,G)$ is a Smarandache semigroup if and only if it is not a group (since $G \subseteq \mathrm{E}(X,G)$), and by [1, Proposition 5.3], equivalently (X,G) is not distal.

Theorem 2. In the transformation semigroup (X,S), if $a \in A \subseteq X$, and S is a Smarandache semigroup, then we have:

1. $\mathrm{E}(X,S)$ is a Smarandache semigroup if and only if (X,S) is not distal;

2. If $\mathrm{F}(a,\mathrm{E}(X))$ is a Smarandache semigroup, then (X,S) is not $a-$distal;

3. If $A \in \overline{\mathrm{M}}(X)$ and $\mathrm{F}(A,\mathrm{E}(X))$ is a Smarandache semigroup, then (X,S) is not $A\underline{^{\overline{\mathrm{M}}}}$distal;

4. If $A \in \overline{\overline{\mathrm{M}}}(X)$ and $\overline{\mathrm{F}}(A,\mathrm{E}(X))$ is a Smarandache semigroup, then (X,S) is not $A\underline{^{\overline{\overline{\mathrm{M}}}}}$distal.

Proof.

1. If (X,S) is not distal, then $\mathrm{E}(X)$ is not a group (see [1, Proposition 5.3]), S is a Smarandache semigroup thus there exists a group $A \subseteq S \subseteq \mathrm{E}(X,S)$, such that $|A| > 1$.

2, 3, 4. Use [2, Theorem 18] and a similar method described for (1).

Lemma 3. In the transformation semigroup (X,S) if I a minimal right ideal of $\mathrm{E}(X)$ (resp. βS), then at least one of the following statements occurs:

1) I is a Smarandache semigroup;

2) I is a group (or equivalently $|\mathrm{J}(I)| = 1$);

3) $I = \mathrm{J}(I)$.

Proof. In the transformation semigroup (X, S) if I is a minimal right ideal of $\mathrm{E}(X)$ (resp. βS), then $\{Iv : v \in \mathrm{J}(I)\}$ is a partition of I to some of its isomorphic subgroups, and for each $v \in J(I)$, Iv is a group with identity v (see [1, Proposition 3.5]).

Theorem 4. In the transformation semigroup (X, S), at least one of the following statements occurs:

1) For each I minimal right ideal of $\mathrm{E}(X)$, I is a Smarandache semigroup;

2) For each I minimal right ideal of $\mathrm{E}(X)$, I is a group (or equivalently $|\mathrm{J}(I)| = 1$);

3) For each I minimal right ideal of $\mathrm{E}(X)$, $I = \mathrm{J}(I)$.

Proof. We distinguish the following steps:

Step 1. If I and J are minimal right ideals of $\mathrm{E}(X, S)$ and $v \in \mathrm{J}(J)$, then $L_v|_I : I \to J$ is bijective, since where exists $u \in \mathrm{J}(I)$ with $uv = u$ and $vu = v$ ([1, Proposition 3.6], thus $L_u|J \circ L_v|_I = \mathrm{id}_I$.

Step 2. If I and J are minimal right ideals of $\mathrm{E}(X, S)$ and I is a Smarandache semigroup, then there exists $H \subseteq I$ such that H is a group with more than one element, suppose u be the identity element of H, then Iu is a subgroup of I with identity u and $Iu \neq u$, choose $v \in \mathrm{J}(J)$ such that $uv = u$ and $vu = v$, $vIu (= L_v(Iu))$ is a subgroup of J with identity v, moreover $|vIu| = |Iu| \geq |H| > 1$ and since $L_v|_I : I \to J$ is bijective and $Iu \neq I$, thus $vIu \neq J$, therefore J is a Smarandache semigroup.

Step 3. If there exists minimal right ideal I of $\mathrm{E}(X)$ such that I is a group, it means $|\mathrm{J}(I)| = 1$, since for each minimal right ideal J of $\mathrm{E}(X)$, we have $|\mathrm{J}(J)| = |\mathrm{J}(I)|$, thus J has a unique idemppotent element say v. $\{Jv\} = \{Jw : w \in \mathrm{J}(J)\}$ is a partition of J to some of its isomorphic subgroups, thus $Jv = J$ is a group.

Step 4. If none of minimal right ideals of $\mathrm{E}(X)$ satisfies items 1 and 2 in Lemma 3, then by Lemma 3, all of them will satisfy item 3 in Lemma 3.

Using Lemma 3, will complete the proof.

Example 5. Let $X = \{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\} \cup \{-1, -2\}$ with the induced topology of $\mathbf{R}$; $S = \{\varphi^i \psi^j : i, j \in \mathbf{N} \cup \{0\}\}$ with discrete topology, where $\psi : X \to X$ and $\varphi : X \to X$ such that:

$$x\psi = \begin{cases} -1 & x = -2 \\ -2 & x = -1 \\ 0 & x \in X - \{-1, -2\} \end{cases}, \quad x\varphi = \begin{cases} \frac{1}{n+1} & x = \frac{1}{n}, n \in \mathbf{N} \\ x & x \in \{0, -1, -2\} \end{cases},$$

and $\psi^0 = \varphi^0 = \mathrm{id}_X$, then S is a Smarandache semigroup(under the composition of maps) since $A = \{\psi, \mathrm{id}_X\}$ is a subgroup of S, moreover in the transformation group (X, S), $\mathrm{E}(X) = S$.

Example 6. If $X = [0, 1]$ with induced topology of $\mathbf{R}$, and $G := \{f : [0, 1] \to [0, 1]$ is a homeomorphism$\}$ with discrete topology, then $\mathrm{E}(X, G)$ is a Smarandache semigroup since it contains G and it is not a group, note that if:

$$xp := \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases},$$

References

[1] Ellis, R., Lectures on topological dynamics, W. A. Benjamin, New York, 1969.

[2] Sabbaghan, M., Ayatollah Zadeh Shirazi, F., $A-$minimal sets and related topics in transformation semigroups (I), International Journal of Mathematics and Mathematical Sciences, **25**(2001), No. 10, 637-654.

[3] Vasantha Kandasamy, W. B., Smarandache Semigroups, American Research Press, Rehoboth, 2002.