# ON $K$-FACTORIALS AND SMARANDACHEIALS

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**Abstract** F. Smarandache defines a $k$-factorial as $n(n-k)(n-2k)\cdots$, terminating when $n-xk$ is positive and $n-(x+1)k$ is 0 or negative. Smarandacheials extend this definition into the negative numbers such that the factorial terminates when $|n-xk|$ is less than or equal to $n$ and $|n-(x+1)k|$ is greater than $n$. This paper looks at some relations between these numbers.

**Keywords:** Smarandache function; Additive Analogue; Mean value formula.

## $k$-factorials

We begin by looking at the $k$-factorial, represented by $k$ exclamation marks after the variable.

The $k$-factorial is merely the full factorial $n!$ with some of the terms omitted.

e.g. the factorial for $n = 8$;

$n! = 8! = 8.7.6.5.4.3.2.1$

If we look at the 2-factorial $n!!$, we have

$n!! = 8.6.4.2$

and we see that $7, 5, 3$ and $1$ are not present.

Similarly the 3-factorial $n!!!$ gives

$n!!! = 8.5.2$

and $7, 6, 4, 2$ and $1$ are not present.

In the first case we have 7!! omitted, so we may write $n!! = n! \mid (n-1)!!$

For the 3-factorial, there are two sequences present, 7!!! and 6!!!, so

$n!!! = n! \mid [(n-1)!!!(n-2)!!!]$

Using the notation $n!_k$ for a $k$-factorial, we can easily obtain the general formula

$$n!_k = \frac{n!}{\prod_{i=1}^{k-1}(n-k)!_k}$$

A PARI/GP program to implement this is

```
{
kfactorial(n,k)=local(result);
result=n!;
for (i=1,k-1,result/=kfactorial(n-i,k));
result
```

```
}
```

although this is highly ineffective and not recommended for use.

To access the k-factorial function use

```
{
kfactorial(n, k)=local(res);
res=vector(n, i,if (i <= k, i, 0));
for (i = k + 1, n,res[i]=i*res[i - k]);
res[n];
}
```

This code stores 1 to $k$ in a vector in positions 1 to $k$. Then each progressive term is calculated form the $k$-th previous entry and the current one. The above code actually calculates $n!_k$ for all $n$ from 1 to $n$.

Even simpler, and the quickest yet is

```
{
kf(n,k)=local(r,c);
c=n
if (c==0,c=k);
r=c;
while (c<n,c+=k;r*=c);
r
}
```

The $r$ variable holds the result, and the $c$ variable is a counter. $c$ is set to $n$ mod $k$, and then incremented until it is $n$. $r$ is multiplied by $c$ at each stage.

We can also see that if $gcd(n, k) = k$, then $n!_k = k^{\frac{n}{k}}(\frac{n}{k})!$, so in this case we have

$$k^{\frac{n}{k}}(\frac{n}{k})!\prod_{i=1}^{k-1}(n-k)!_k = n!$$

If $gcd(n, k)! = k$, then the $k$-factorial seems more difficult to define. We address this problem shortly.

**Smarandacheials**

In extending the $k$-factorial to the negative integers, we need to further define the parameters involved.

If we let $n$ be the starting number, and $k$ be the decrease, then we also need to define $m$ as $n$ mod $k$, and then $m' = k - m$.

If $m$ is greater than or equal to $m'$, we can see that $SM(n, k) = \pm[n!_k * (n-(m-m'))!_k]$.

If $m$ is less than $m'$, then we have $SM(n, k) = \pm[n!_k*(n-(m-m')-k)!_k]$.

The plus/minus sign is not known yet - this is developed later in this paper.

This result follows because $m$ represents the last integer from $n!_k$, and so $k - m$ will be the first negative integer from$(n - x)!_k$, and so we determine $x$.

If $m$ is greater than or equal to $m'$, then the difference $n - (n - x)$ must be the difference between $m$ and $m'$, so $x = m - m'$.

If $m < m'$, then we have a problem. The basic idea still works, however the negative factorial will rise to a higher level than the original $n$, and this is not allowed. So the adjustment from subtracting $k$ cuts the last negative integer out of the equation.

To combine these, define $m^*$ as the smallest positive value of $(m - m')$ mod $k$, and now we may write;

$$SM(n,k) = \pm[n!_k^*(n - m^*)!_k].$$

For an example, consider $SM(13,5)$. Then $m = 3$ and $m' = 2$, and $m^* = (3-2) \bmod 5 = 1$, so we get;

$$\begin{aligned} SM(13,5) &= \pm 13!_5{}^*(13-1)!_5 \\ &= \pm 13!_5{}^*(12)!_5 \\ &= \pm 13.8.3^*12.7.2 \end{aligned}$$

However, for$SM(12,5), m^* = (2-3) \bmod 5 = 4$

$$\begin{aligned} &= \pm 12!_5{}^*(12-4)!_5 \\ &= \pm 12!_5{}^*8!_5 \\ &= \pm 12.7.2^*8.3 \end{aligned}$$

The sign is then simply $(-1)^\wedge$(number of terms in second Smarandacheial).

**Number of terms in $n!_k$**

Given $n!_k$, the expansion of the expression is;

$n(n-k)\cdots(n-ak)$

So there are $a+1$ terms.

Using a simple example, e.g. for $k = 5$, we can construct a table of the number of terms;

| n | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|
| no. of terms | 1 | 2 | 2 | 2 | 2 | 2 | 3 |

| n | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|
| no. of terms | 3 | 3 | 3 | 3 | 4 | 4 | 4 |

From this we see that there are ceil($\frac{n}{k}$) terms.

Therefore a full expression for the Smarandacheial function is

$$SM(n,k) = (-1)^\wedge \left[\frac{n-m^*}{k}\right] n!_k(n-m^*)!_k$$

**$n!_k$ for** gcd$(n,k) < k$

In this case, we have no easy relation. We can however spot an interesting and deep relation with these $k$-factorials - their relation to a neighbouring $n!_k$ with $gcd(n,k) = k$.

To demonstrate this connection, we will examine $15!_5$.

This is $15.10.5 = 750$.

Now $16!_5$ is $16.11.6.1$. There seems to be nothing else we can do.
However, we can write this as;

$$\left(\frac{16}{15}\right)\left(\frac{11}{10}\right)\left(\frac{6}{5}\right)15!_5$$

Still nothing, but then we see that $\frac{16}{15} = 1 + \frac{1}{15}$, and so on, and so we get;

$$\left(1+\frac{1}{15}\right)\left(1+\frac{1}{10}\right)\left(1+\frac{1}{5}\right)15!_5$$

If we expand the brackets, hey presto (I have skipped a few steps here)

$$16!_5 = 15!_5 + 15.10 + 15.5 + 10.5 + 15 + 10 + 5 + 1$$

This is generalizable into

$$(n+x)!_k = \left[n!_k + \sum_{d\in S}\frac{x^{d_i}n!_k}{d}\right]x$$

where $x$ is less than $k$, $S$ is the distinct power set of components of $n!_k$ (e.g. for $15!_5, S = 15, 10, 5$), and $d_i$ is the number of elements of $S$ involved in $d$.
For example, $18!_5$ gives

$$18!_5 = [15!_5 + 3(15.10 + 15.5 + 10.5) + 9(15 + 10 + 5) + 27]^{*}3$$

$18!_6 = 5616$, and the sum on the RHS is
$750 + 825 + 270 + 27 = 1872$, and $1872^{*}3 = 5616$.

## References

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