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Smarandache fantastic ideals of Smarandache BCI-algebras

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Abstract The notion of Smarandache fantastic ideals is introduced, examples are given, and related properties are investigated. Relations among Q-Smarandache fresh ideals, Q-Smarandache clean ideals and Q-Smarandache fantastic ideals are given. A characterization of a Q-Smarandache fantastic ideal is provided. The extension property for Q-Smarandache fantastic ideals is established.

Keywords Smarandache BCI-algebra, Smarandache fresh ideal, Smarandache clean ideal, Smarandache fantastic ideal

1. Introduction

Generally, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [5], W.B.Vasantha Kandasamy studied the concept of Smarandache groupoids, sub-groupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R.Padilla [4]. It will be very interesting to study the Smarandache structure in BCK/BCI-algebras. In [1], Y.B.Jun discussed the Smarandache structure in BCI-algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals, and investigated some related properties. Also, he studied Smarandache ideal structures in Smarandache BCI-algebras. He introduced the notion of Smarandache fresh ideals and Smarandache clean ideals in Smarandache BCI-algebras, and investigated its useful properties. He gave relations between Q-Smarandache fresh ideals and Q-Smarandache clean ideals, and established extension properties for Q-Smarandache fresh ideals and Q-Smarandache clean ideals (see [2]). In this paper, we introduce the notion of Q-Smarandache fantastic ideals, and investigate its properties. We give relations among Q-Smarandache fresh ideals, Q-Smarandache clean ideals and Q-Smarandache fantastic ideals. We also provide a characterization of a Q-Smarandache fantastic ideal. We finally establish the extension property for Q-Smarandache fantastic ideals.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(a1) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,

(a2) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,

(a3) $(\forall x \in X)$ $(x * x = 0)$,

(a4) $(\forall x, y \in X)$ $(x * y = 0,\ y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(a5) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a BCK-algebra. We can define a partial order $\leq$ on X by $x \leq y \iff x * y = 0$. Every BCI-algebra X has the following properties:

(b1) $(\forall x \in X)$ $(x * 0 = x)$.

(b2) $(\forall x, y, z \in X)$ $((x * y) * z = (x * z) * y)$.

(b3) $(\forall x, y, z \in X)$ $(x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$.

(b4) $(\forall x, y \in X)$ $(x * (x * (x * y)) = x * y)$.

A Smrandache BCI-algebra [1] is defined to be a BCI-algebra X in which there exists a proper subset Q of X such that

(s1) $0 \in Q$ and $|Q| \geq 2$,

(s2) Q is a BCK-algebra under the operation of X.

3. Smarandache Fantastic Ideals

In what follows, let X and Q denote a Smarandache BCI-algebra and a BCK-algebra which is properly contained in X, respectively.

Definition 3.1. [1] A nonempty subset I of X is called a Smarandache ideal of X related to Q (or briefly, Q-Smarandache ideal of X) if it satisfies:

(c1) $0 \in I$,

(c2) $(\forall x \in Q)$ $(\forall y \in I)$ $(x * y \in I \Rightarrow x \in I)$.

If I is a ideal of X related to every BCK-algebra contained in X, we simply say that I is a Smarandache ideal of X.

Definition 3.2. [2] A nonempty subset I of X is called a Smarandache fresh ideal of X related to Q (or briefly, Q-Smarandache fresh ideal of X) if it satisfies the condition (c1) and (c3) $(\forall x, y, z \in Q)\ ((x * y) * z \in I,\ y * z \in I \Rightarrow x * z \in I)$.

Lemma 3.3. [2] If I is a Q-Smarandache fresh ideal of X, then

(i) $(\forall x, y \in Q)\ ((x * y) * y \in I \Rightarrow x * y \in I)$.

(ii) $(\forall x, y, z \in Q)\ ((x * y) * z \in I \Rightarrow (x * z) * (y * z) \in I)$.

Definition 3.4. [2] A nonempty subset I of X is called a Smarandache clean ideal of X related to Q (or briefly, Q-Smarandache clean ideal of X) if it satisfies the condition (c1) and

(c4) $(\forall x, y \in Q)\ (\forall z \in I)\ ((x * (y * x)) * z \in I \Rightarrow x \in I)$.

Lemma 3.5. [2] Every Q-Smarandache clean ideal is a Q-Smarandache fresh ideal.

Lemma 3.6. Let I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache clean ideal of $X \iff I$ satisfies the following condition:

$$(\forall x, y \in Q)\ (x * (y * x) \in I \Rightarrow x \in I). \tag{1}$$

Proof. Suppose that I satisfies the condition (1) and suppose that $(x * (y * x)) * z \in I$ for all $x, y \in Q$ and $z \in I$. Then $x * (y * x) \in I$ by (c2), and so $x \in I$ by (1). Conversely assume that I is a Q-Smarandache clean ideal of X and let $x, y \in Q$ be such that $x * (y * x) \in I$. Since $0 \in I$, it follows from (b1) that $(x * (y * x)) * 0 = x * (y * x) \in I$ so from (c4) that $x \in I$. This completes the proof.

Definition 3.7. A nonempty subset I of X is called a Smarandache fantastic ideal of X related to Q (or briefly, Q-Smarandache fantastic ideal of X) if it satisfies the condition (c1) and

(c5) $(\forall x, y \in Q)\ (\forall z \in I)\ ((x * y) * z \in I \Rightarrow x * (y * (y * x)) \in I)$.

Example 3.8. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	1	0	1	5
2	2	2	0	2	0	5
3	3	1	3	0	3	5
4	4	4	4	4	0	5
5	5	5	5	5	5	0

Table 3.1

Then $(X; *, 0)$ is a Smarandache BCI-algebra. Note that $Q = \{0, 1, 2, 3, 4\}$ is a BCK-algebra which is properly contained in X. It is easily checked that subsets $I_1 = \{0, 2\}$ and $I_2 = \{0, 2, 4\}$ are Q-Smarandache fantastic ideals of X, but not Q-Smarandache fresh ideals. A subset $I_3 = \{0, 1, 3\}$ is a Q-Smarandache fresh ideal, but not a Q-Smarandache fantastic ideal since $(2 * 4) * 3 = 0 \in I_3$ and $2 * (4 * (4 * 2)) = 2 \notin I_3$.

The example above suggests that a Q-Smarandache fantastic ideal need not be a Q-Smarandache fresh ideal, and a Q-Smarandache fresh ideal may not be a Q-Smarandache fantastic ideal.

Theorem 3.9. Let Q_1 and Q_2 be BCK-algebras which are properly contained in X such that $Q_1 \subset Q_2$. Then every Q_2-Smarandache fantastic ideal is a Q_1-Smarandache fantastic ideal of X.

Proof. Straightforward.

The converse of Theorem 3.9 is not true in general as seen in the following example.

Example 3.10. Consider the Smarandache BCI-algebra X described in Example 3.8. Note that $Q_1 := \{0, 2, 4\}$ and $Q_2 := \{0, 1, 2, 3, 4\}$ are BCK-algebras which are properly contained in X and $Q_1 \subset Q_2$. Then $I := \{0, 1, 3\}$ is a Q_1-Smarandache fantastic ideal, but not a Q_2-Smarandache fantastic ideal of X.

Theorem 3.11. Every Q-Smarandache fantastic ideal is a Q-Smarandache ideal.

Proof. Let I be a Q-Smarandache fantastic ideal of X and assume that $x * z \in I$ for all $x \in Q$ and $z \in I$. Using (b1), we get $(x*0)*z = x*z \in I$. Since $x \in Q$ and Q is a BCK-algebra, it follows from (a5), (b1) and (c5) that $x = x * (0 * (0 * x)) \in I$ so that I is a Q-Smarandache ideal of X.

As seen in Example 3.8, the converse of Theorem 3.11 is not true in general.

Theorem 3.12. Let I be a Q-Smarandache ideal of X. Then I is a Q-Smarandache fantastic ideal of $X \iff$ it satisfies the following implication:

$$(\forall x, y \in Q)\,(x * y \in I \Rightarrow x * (y * (y * x)) \in I). \tag{2}$$

Proof. Assume that I is a Q-Smarandache fantastic ideal of X and let $x, y \in Q$ be such that $x * y \in I$. Using (b1), we have $(x * y) * 0 = x * y \in I$ and $0 \in I$. It follows from (c5) that $x * (y * (y * x)) \in I$. Conversely suppose that I satisfies the condition (2). Assume that $(x * y) * z \in I$ for all $x, y \in Q$ and $z \in I$. Then $x * y \in I$ by (c2), and hence $x * (y * (y * x)) \in I$ by (2). This completes the proof.

Theorem 3.13. Let I be a nonempty subset of X. Then I is a Q-Smarandache clean ideal of $X \iff I$ is both a Q-Smarandache fresh ideal and a Q-Smarandache fantastic ideal of X.

Proof. Assume that I is a Q-Smarandache clean ideal of X. Then I is a Q-Smarandache fresh ideal of X (see Lemma 3.5). Suppose that $x * y \in I$ for all $x, y \in Q$. Since Q is a BCK-algebra, we have

$$(x * (y * (y * x))) * x = (x * x) * (y * (y * x)) = 0 * (y * (y * x)) = 0,$$

and so $(y * x) * (y * (x * (y * (y * x)))) = 0$, that is, $y * x \leq y * (x * (y * (y * x)))$. It follows from (b3), (b2) and (a1) that

$$\begin{aligned} &(x * (y * (y * x))) * (y * (x * (y * (y * x)))) \\ &\leq (x * (y * (y * x))) * (y * x) \\ &= (x * (y * x)) * (y * (y * x)) \leq x * y, \end{aligned}$$

that is, $((x*(y*(y*x)))*(y*(x*(y*(y*x)))))*(x*y) = 0 \in I$. Since $x*y \in I$, it follows from (c2) that $(x*(y*(y*x)))*(y*(x*(y*(y*x)))) \in I$, so from Lemma 3.6 that $x*(y*(y*x)) \in I$. Using Theorem 3.12, we know that I is a Q-Smarandache fantastic ideal of X.

Conversely, suppose that I is both a Q-Smarandache fresh ideal and a Q-Smarandache fantastic ideal of X. Let $x, y \in Q$ be such that $x * (y * x) \in I$. Since

$$((y * (y * x)) * (y * x)) * (x * (y * x)) = 0 \in I,$$

we get $(y * (y * x)) * (y * x) \in I$ by (c2). Since I is a Q-Smarandache fresh ideal, it follows from Lemma 3.3(i) that $y * (y * x) \in I$ so from (c2) that $x * y \in I$ since $(x * y) * (y * (y * x)) = 0 \in I$. Since I is a Q-Smarandache fantastic ideal, we obtain $x * (y * (y * x)) \in I$ by (2), and so $x \in I$ by (c2). Therefore I is a Q-Smarandache clean ideal of X by Lemma 3.6.

Theorem 3.14. (Extension Property) Let I and J be Q-Smarandache ideals of X and $I \subset J \subset Q$. If I is a Q-Smarandache fantastic ideal of X, then so is J.

Proof. Assume that $x * y \in J$ for all $x, y \in Q$. Since

$$(x * (x * y)) * y = (x * y) * (x * y) = 0 \in I,$$

it follows from (b2) and (2) that

$$(x * (y * (y * (x * (x * y))))) * (x * y) = (x * (x * y)) * (y * (y * (x * (x * y)))) \in I \subset J,$$

so from (c2) that $x * (y * (y * (x * (x * y)))) \in J$. Since $x, y \in Q$ and Q is a BCK-algebra, we get $(x * (y * (y * x))) * (x * (y * (y * (x * (x * y))))) = 0 \in J$, by using (a1) repeatedly. Since J is a Q-Smarandache ideal, we conclude that $x * (y * (y * x)) \in J$. Hence J is a Q-Smarandache fantastic ideal of X by Theorem 3.12.

References

[1] Y. B. Jun, Smarandache BCI-algebras, Sci. Math. Jpn., **62**(2005), No. 1, 137–142, **e2005**, 271–276.

[2] Y. B. Jun, Smarandache fresh and clean ideals of Smarandache BCI-algebras, Kyungpook Math. J., **46**(2006), 409–416.

[3] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa Co., Seoul, Korea, 1994.

[4] R. Padilla, Smarandache algebraic structures, Bull. Pure Appl. Sci., Delhi, **17E**(1998), No. 1, 119–121, http://www.gallup.unm.edu/∼Smarandache/alg-s-tx.txt.

[5] W. B. Vasantha Kandasamy, Smarandache groupoids, http://www.gallup.unm.edu/∼smarandache/Groupoids.pdf.