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On the F.Smarandache LCM function $SL(n)$

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The main purpose of this paper is using the elementary methods to study the mean value distribution property of $(P(n) - p(n))SL(n)$, and give an interesting asymptotic formula for it.

Keywords $SL(n)$ function, mean value distribution, asymptotic formula.

§1. Introduction and Result

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. From the definition of $SL(n)$ we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}. \tag{1}$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [1], [2] and [3]. For example, Murthy [1] porved that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ be the F.Smarandache function. That is, $S(n) = \min\{m:\ n|m!,\ m \in N\}$. Simultaneously, Murthy [1] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{2}$$

Le Maohua [2] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \ \ \texttt{or} \ \ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where $p_1,\ p_2,\ \cdots,\ p_r,\ p$ are distinct primes and $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_r$ are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1,\ 2,\ \cdots,\ r$.

Zhongtian Lv [3] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

Jianbin Chen [4] studied the value distribution properties of $SL(n)$, and proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(SL(n) - P(n))^2 = \frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

Xiaoyan Li [5] studied the mean value properties of $P(n)SL(n)$ and $p(n)SL(n)$, and give two sharper asymptotic formulas for them, where $p(n)$ denotes the smallest prime divisor of n.

Yanrong Xue [6] defined another new function $SL^*(n)$ as follows: $SL^*(1) = 1$, and if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL^*(n) = \min\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}, \tag{3}$$

where $p_1 < p_2 < \ \cdots < p_r$ are primes.

It is clear that function $SL^*(n)$ is the dual function of $SL(n)$. So it has close relationship with $SL(n)$. About its elementary property of the function $SL^*(n)$, Yanrong Xue [6] proved the following conclusion:

For any positive integer n, there is no any positive integer $n > 1$ such that

$$\sum_{d|n}\frac{1}{SL^*(d)}$$

is an positive integer, where $\sum\limits_{d|n}$ denotes the summation over all positive divisors of n.

In this paper, we shall study the value distribution properties of $(P(n) - p(n))SL(n)$, and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x > 1$ and any positive integer k, we have the asymptotic formula

$$\sum_{n\leq x}(P(n) - p(n))SL(n) = \zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, $b_1 = \dfrac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem directly. For any positive integer $n > 1$, we consider the following cases:

A: $n = n_1\cdot p$, $n_1 \leq p$, and $SL(n) = p$;

B: $n = n_2 \cdot p$, $n_2 > p$, and $SL(n) = p$;

C: $n = m \cdot p^\alpha$, $\alpha \geq 2$, and $SL(n) = p^\alpha$;

Now, for any positive integer $n > 1$, we consider the summation:

$$\sum_{n\leq x}(P(n)-p(n))SL(n).$$

It is clear that if $n \in A$, then from (1) we know that $SL(n) = p$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [7]) and the Prime Theorem (See Theorem 3.2 of [8]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1, 2, \ldots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{\substack{n\leq x\\ n\in A}}(P(n)-p(n))SL(n) &= \sum_{\substack{n\leq x\\ n=n_1\cdot p, n_1\leq p\\ SL(n)=p}} (P(n)-p(n))SL(n)\\
&= \sum_{n_1\leq\sqrt{x}}\sum_{n_1\leq p\leq \frac{x}{n_1}} (P(n_1\cdot p)-p(n_1\cdot p))p\\
&= \sum_{n_1\leq\sqrt{x}}\sum_{n_1\leq p\leq \frac{x}{n_1}} (p-p(n_1))p\\
&= \sum_{n_1\leq\sqrt{x}}\sum_{n_1\leq p\leq \frac{x}{n_1}} p^2 - \sum_{n_1\leq\sqrt{x}}\sum_{n_1\leq p\leq \frac{x}{n_1}} p(n_1)p, \qquad (4)
\end{aligned}$$

while

$$\begin{aligned}
\sum_{n_1\leq\sqrt{x}}\sum_{n_1\leq p\leq \frac{x}{n_1}} p^2 &= \sum_{n_1\leq\sqrt{x}}\left[\frac{x^2}{n_1^2}\pi\left(\frac{x}{n_1}\right) - \int_{n_1}^{\frac{x}{n_1}} 2y\pi(y)dy + O\left(n_1^3\right)\right]\\
&= \sum_{n_1\leq\sqrt{x}}\left[\frac{x^3}{n_1^3}\sum_{i=1}^{k}\frac{b_i}{\ln^i \frac{x}{n_1}} + O\left(\frac{x^3}{n_1^3\cdot\ln^{k+1}\frac{x}{n_1}}\right)\right]\\
&= \zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right), \qquad (5)
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, $b_1 = \frac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

Note that $p(n_1) \leq n_1$, we have

$$\begin{aligned}
\sum_{n_1\leq\sqrt{x}}\sum_{n_1\leq p\leq \frac{x}{n_1}} p(n_1)p &= \sum_{n_1\leq\sqrt{x}} p(n_1) \sum_{n_1\leq p\leq \frac{x}{n_1}} p\\
&= \sum_{n_1\leq\sqrt{x}} p(n_1)\left[\frac{x}{n_1}\pi\left(\frac{x}{n_1}\right) - \int_{n_1}^{\frac{x}{n_1}}\pi(y)dy + O\left(n_1^2\right)\right]\\
&\ll \sum_{n_1\leq\sqrt{x}} p(n_1)\cdot\frac{x^2}{n_1^2\ln x} \ll \sum_{n_1\leq\sqrt{x}}\frac{x^2}{n_1\ln x} = O(x^2). \qquad (6)
\end{aligned}$$

From (4), (5)and (6) we have

$$\sum_{\substack{n\le x\\ n\in A}}(P(n)-p(n))SL(n)=\zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right), \tag{7}$$

where $b_1=\dfrac{1}{3}$, b_i $(i=2,3,\cdots,k)$ are computable constants.

If $n\in B$, $SL(n)=p$, then by the Abel's summation formula and the Prime Theorem, we can deduce the following:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in B}}(P(n)-p(n))SL(n) &= \sum_{\substack{n\le x\\ n=n_2\cdot p,n_2>p\\ SL(n)=p}}(P(n)-p(n))SL(n)\\ &= \sum_{\substack{n_2\cdot p\le x\\ n_2>p}}(p-p(n_2))p\\ &\ll \sum_{\substack{n_2\cdot p\le x\\ n_2>p}}p^2=\sum_{p<\sqrt{x}}\sum_{p<n_2\le\frac{x}{p}}p^2\\ &< \sum_{p<\sqrt{x}}\frac{x}{p}\cdot p^2=\sum_{p<\sqrt{x}}x\cdot p\\ &= x\sum_{p<\sqrt{x}}p\ll x^2. \end{aligned}\tag{8}$$

If $n\in C$, then $SL(n)=p^\alpha$, $\alpha\ge 2$. Therefore, using the Abel's summation formula and the Prime Theorem, we can obtain:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in C}}(P(n)-p(n))SL(n) &= \sum_{\substack{n\le x\\ n=m\cdot p^\alpha,\alpha\ge2\\ SL(n)=p^\alpha}}(P(n)-p(n))SL(n)\\ &= \sum_{\substack{m\cdot p^\alpha\le x\\ \alpha\ge2}}(P(m\cdot p^\alpha)-p(m\cdot p^\alpha))p^\alpha\\ &\ll \sum_{\substack{m\cdot p^\alpha\le x\\ \alpha\ge2}}p^{2\alpha}\ll\sum_{\substack{p^\alpha\le x\\ \alpha\ge2}}\sum_{m\le\frac{x}{p^\alpha}}p^{2\alpha}\\ &\ll \sum_{\substack{p^\alpha\le x\\ \alpha\ge2}}\frac{x}{p^\alpha}\cdot p^{2\alpha}=\sum_{\substack{p^\alpha\le x\\ \alpha\ge2}}x\cdot p^\alpha\\ &= x\sum_{\substack{p^\alpha\le x\\ \alpha\ge2}}p^\alpha=x\sum_{\substack{p\le x^{\frac{1}{\alpha}}\\ \alpha\ge2}}p^\alpha\ll x^{\frac{5}{2}}. \end{aligned}\tag{9}$$

Now, combining (7), (8) and (9) we may immediately obtain the fowllowing asymptotic formula:

$$\sum_{n\leq x}(P(n)-p(n))SL(n)=\zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),$$

where $P(n)$ and $p(n)$ denote the largest and smallest prime divisor of n respectively, $\zeta(s)$ is the Riemann zeta-function, $b_1=\frac{1}{3}$, b_i $(i=2,3,\cdots,k)$ are computable constants.

This completes the proof of Theorem.

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