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The Forcing Domination Number of Hamiltonian Cubic Graphs

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Abstract: A set of vertices S in a graph G is called to be a Smarandachely dominating k-set, if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G. The Smarandachely domination number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely dominating set of G. For abbreviation, we denote $\gamma_1(G)$ by $\gamma(G)$. In 1996, Reed proved that the domination number $\gamma(G)$ of every n-vertex graph G with minimum degree at least 3 is at most $3n/8$. Also, he conjectured that $\gamma(H) \geq \lceil n/3 \rceil$ for every connected 3-regular n-vertex graph H. In [**?**], the authors presented a sequence of Hamiltonian cubic graphs whose domination numbers are sharp and in this paper we study forcing domination number for those graphs.

Key Words: Smarandachely dominating k-set, dominating set, forcing domination number, Hamiltonian cubic graph.

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§1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [12] for terminology in graph theory.

Let G be a graph, with n vertices and e edges. Let $N(v)$ be the set of neighbors of a vertex v and $N[v] = N(v) \cup \{v\}$. Let $d(v) = |N(v)|$ be the degree of v. A graph G is $r-$regular if $d(v) = r$ for all v. Particularly, if $r = 3$ then G is called a cubic graph. A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is called to be a *Smarandachely dominating k-set*, if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a *dominating set* of G. The *Smarandachely domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely dominating set of G. For abbreviation, we denote $\gamma_1(G)$ by $\gamma(G)$. A subset F of a minimum dominating set S is a *forcing subset* for S if S is the unique minimum dominating set containing F. The *forcing domination number* $f(G, \gamma)$ of S is the minimum cardinality among the forcing subsets of S , and the forcing domination number $f(G, \gamma)$ of G is the minimum forcing domination number among

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the minimum dominating sets of G ([1], [2], [5]-[7]). For every graph G, $f(G,\gamma) \leq \gamma(G)$. Also The forcing domination number of several classes of graphs are determined, including complete multipartite graphs, paths, cycles, ladders and prisms. The forcing domination number of the cartesian product G of k copies of the cycle C_{2k+1} is studied.

The problem of finding the domination number of a graph is NP-hard, even when restricted to cubic graphs. One simple heuristic is the greedy algorithm, ([11]). Let d_g be the size of the dominating set returned by the greedy algorithm. In 1991 Parekh [9] showed that $d_g \leq n + 1 - \sqrt{2e+1}$. Also, some bounds have been discovered on $\gamma(G)$ for cubic graphs. Reed [10] proved that $\gamma(G) \leq \frac{3}{8}n$. He conjectured that $\gamma(H) \geq \lceil \frac{n}{3} \rceil$ for every connected 3-regular (cubic) n-vertex graph H. Reed's conjecture is obviously true for Hamiltonian cubic graphs. Fisher et al. [3]-[4] repeated this result and showed that if G has girth at least 5 then $\gamma(G) \leq \frac{5}{14}n$. In the light of these bounds on γ, in 2004 Seager considered bounds on d_g for cubic graphs and showed that ([11]):

For any graph of order n, $\left\lceil \frac{n}{1+\Delta G} \right\rceil \leq \gamma(G)$ (see [4]) and for a cubic graph G, $d_g \leq \frac{4}{9}n$.

In this paper, we would like to study the forcing domination number for Hamiltonian cubic graphs. In [8], the authors showed that:

Lemma A. If $r \equiv 2$ or 3 (mod 4), then $\gamma\left(G'\right) = \gamma(G)$.

Lemma B. If $r \equiv 0$ or 1 (mod 4), then $\gamma\left(G'\right) = \gamma(G) - 1$.

Theorem C. If $r \equiv 1$ (mod 4), then $\gamma(G_0) = m\lceil \frac{n}{4} \rceil - \lceil \frac{m}{3} \rceil$.

§2. Forcing domination number

Remark 2.1 Let $G = (V,E)$ be the graph with $V = \{v_1, v_2, ..., v_n\}$ for $n = 2r$ and $E = \{v_i v_j | \; |i-j| = 1 \text{ or } r\}$. So G has two vertices v_1 and v_n of degree two and $n-2$ vertices of degree three. By the graph G is the graph described in Fig.1.

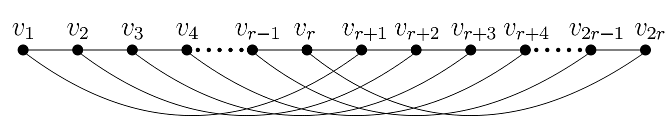

Fig.1. The graph G.

For the following we put $N_p[x] = \{z | \; z \text{ is only dominated by } x\} \cup \{x\}$.

Remark 2.2 Suppose that the graphs G' and G'' are two induced subgraphs of G such that $V(G') = V(G) - \{v_1, v_n\}$ and $V(G'') = V(G) - \{v_1\}$ (or $V(G'') = V(G) - \{v_{2r}\}$).

Remark 2.3 Let G_0 be a graph of order mn that $n = 2r$, $V(G_0) = \{v_{11}, v_{12}, ..., v_{1n}, v_{21}, v_{22}, ..., v_{2n}, ..., v_{m1}, v_{m2} ..., v_{mn}\}$ and $E = \cup_{i=1}^{m}\{v_{ij}v_{il} | \; |j-l| = 1 \text{ or } r\} \cup \{v_{in}v_{(i+1)1} | \; i = 1, 2, ..., m-1\} \cup \{v_{11}v_{mn}\}$. By the graph G_0 is 3-regular graph. Suppose that the graph G_i

is an induced subgraph of G_0 with the vertices $v_{i1}, v_{i1}, ..., v_{in}$. By the graph G_0 is the graph described in Fig. 2.

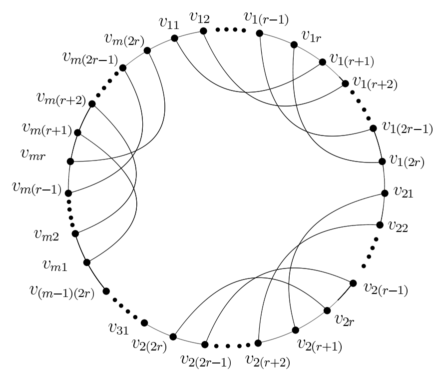

Fig. 2. The graph G_0.

Proposition 2.4 *If* $r \equiv 0$ *(mod 4), then* $f(G, \gamma) \leq 2$, *otherwise* $f(G, \gamma) = 1$.

proof First we suppose that $r \equiv 1$ (mod 4). It is easy to see that $f(G, \gamma) > 0$, because G has at least two minimum dominating set. Suppose $F = \{v_1\} \subset S$ where S is a minimum dominating set. Since $\gamma(G) = 2\lfloor r/4 \rfloor + 1$, for two vertices v_x and v_y in S, $|N[v_x] \cup N[v_y]| \geq 6$. This implies that $\{v_2, v_{r+1}\} \cap S = \emptyset$, then $v_{r+3} \in S$. A same argument shows that $v_5 \in S$. Thus S must be contains $\{v_{r+7}, v_9, ..., v_{2r-2}, v_r\}$, therefore $f(G, \gamma) = 1$.

If $r \equiv 2$ (mod 4), we consider $S = \{v_2, v_6, v_{10}, ..., v_r, v_{r+4}, v_{r+8}, ..., v_{2r-6}, v_{2r-2}\}$. Assign the set $F = \{v_2\}$ then it follows $f(G, \gamma) \leq 1$, because $|N_p[x]| = 4$ to each vertex $x \in S$. On the other hand since G has at least two minimum dominating set. Hence $f(G, \gamma) = 1$.

If $r \equiv 3$ (mod 4), for $S = \{v_1, v_5, v_9, ..., v_{r-2}, v_{r+3}, v_{r+7}, ..., v_{2r-4}, v_{2r}\}$, the set $F = \{v_1\}$ shows that $f(G, \gamma) \leq 1$. Further, since G has at least two minimum dominating set, then it follows $f(G, \gamma) = 1$.

Finally let $r \equiv 0$ (mod 4), we consider $S = \{v_1, v_5, v_9, ..., v_{r-3}, v_{r+1}, v_{r+3}, v_{r+7}, ..., v_{2r-5}, v_{2r-1}\}$. If $F = \{v_1, v_{r+1}\}$, a simple verification shows that $f(G, \gamma) \leq 2$. □

Proposition 2.5 *If* $r \equiv 1$ *(mod 4) then* $f\left(G', \gamma\right) = 0$.

Proof By Lemma B, we have $\gamma\left(G'\right) = 2\lfloor r/4 \rfloor$. Now, we suppose that S is an arbitrary minimum dominating set for G'. Obviously for each vertex $v_x \in S$, $|N_p[v_x]| = 4$, so $\{v_{r-1}, v_{r+2}\} \subset S$. But $\{v_{2r-2}, v_{r-2}\} \cap S = \emptyset$ therefore $v_{2r-3} \in S$. Thus S must be contains $\{v_{r-5}, v_{r-9}, ..., v_{r+10}, v_{r+6}\}$, then S is uniquely determined and it follows that $f\left(G', \gamma\right) = 0$.□

Proposition 2.6 *If* $r \equiv 0$ *(mod 4) then* $f\left(G'', \gamma\right) = 0$.

Proof Let $r \equiv 0$ (mod 4) and S be an arbitrary minimum dominating set for G'' with $V(G'') = V(G) - \{v_1\}$. If $\{v_{2r}, v_{2r-1}\} \cap S \neq \emptyset$. Without loss of generality, we assume that $v_{2r} \in S$ then S must be contains $\{v_{r+2}, v_{r-2}, v_{r-6}, ..., v_{10}, v_6, v_{2r-4}, v_{2r-8}, ..., v_{r+8}\}$. On the other hand by Lemma B, $\gamma\left(G''\right) = 2\lfloor r/4 \rfloor$ (Note that by Proof of Lemma B one can see

$\gamma(G^{'}) = \gamma(G^{''})$ where $r \equiv 0 \pmod 4$). So the vertices v_3, v_4, v_{r+4} and v_{r+5} must be dominated by one vertex and this is impossible. Thus necessarily $v_r \in S$, but $\{v_{r-1}, v_{2r-1}\} \cap S = \emptyset$ which implies $v_{2r-2} \in S$. Finally the remaining non-dominated vertices $\{v_{r+1}, v_{r+2}, v_2\}$ is just dominated by v_{r+2}. Therefore the set $S = \{v_4, v_8, ..., v_{r-4}, v_r, v_{r+2}, v_{r+6}, ..., v_{2r-2}\}$ is uniquely determined which implies $f\left(G^{''}, \gamma\right) = 0$. □

§3. Main Results

Theorem 3.1 *If $r \equiv 2$ or $3 \pmod 4$, then $f(G_0, \gamma) = m$.*

Proof Let $r \equiv 2 \pmod 4$ and S be a minimum dominating set for G_0. If there exists $i \in \{1, 2, ..., m\}$ such that $S \cap \{v_{i1}, v_{in}\} \neq \emptyset$ then it implies $|S \cap G_i| > 2\lfloor r/4 \rfloor + 1$. Moreover $\gamma(G_0) = m(2\lfloor r/4 \rfloor + 1)$. From this it immediately follows that there exists $j \in \{1, 2, ..., m\} - \{i\}$ such that $|S \cap G_j| < 2\lfloor r/4 \rfloor + 1$ and this is contrary to Lemma A. Hence $S \cap \{v_{i1}, v_{in}\} = \emptyset$ for $1 \leq i \leq m$. On the other hand $f(G_i, \gamma) = 1$ for $1 \leq i \leq m$ which implies $f(G_0, \gamma) = m$.

Now we suppose that $r \equiv 3 \pmod 4$ and S is minimum dominating set for G_0, such that $F = \{v_{i1} | 1 \leq i \leq m\} \subset S$. Since $v_{i1} \in S$ and $\gamma(G_0) = 2\lfloor r/4 \rfloor + 2$ then $\{v_{i2}, v_{i3}\} \cap S = \emptyset$ and this implies $v_{i(r+3)} \in S$. With similar description, we have $\{v_{i5}, v_{i9}, ..., v_{i(r-2)}, v_{i(r+6)}, v_{i(r+11)}, ..., v_{i(2r-4)}\} \subset S$. But for the remaining non-dominated vertices $v_{ir}, v_{i(2r)}$ and $v_{i(2r-1)}$ necessarily implies that $v_{i(2r)} \in S$. Hence S is the unique minimum dominating set containing F. Thus $f(G_0, \gamma) \leq m$. A trivial verification shows that $f\left(G^{'}, \gamma\right), f\left(G^{''}, \gamma\right) \geq 1$ for $i \in \{1, 2, ..., m\}$, therefore $f(G_0, \gamma) = m$. □

Theorem 3.2 $f(G_0, \gamma) = \begin{cases} 1 & \text{if } m \equiv 0 \pmod 3 \\ 2 & \text{otherwise} \end{cases}$ *for $r \equiv 1 \pmod 4$.*

Proof If $m \equiv 0 \pmod 3$, we suppose that $F = \{v_{1n}\} \subset S$ and S is a minimum dominating set for G_0. By Theorem C, we have $\gamma(G_0) = m\lceil n/4 \rceil - \lfloor m/3 \rfloor$, then $v_{3,1} \in S$. Here, we use the proof of Propositions 4 and 5. From this the sets $S \cap V(G_1)$, $S \cap V(G_2)$, $S \cap V(G_3)$ uniquely characterize. By continuing this process the set S uniquely obtain, then $f(G_0, \gamma) = 1$.

If $m \equiv 1$ or $2 \pmod 3$, then the set $F = \{v_{1n}, v_{mn}\}$ uniquely characterize the minimum dominating set for G_0, therefore $f(G_0, \gamma) = 2$. □

Theorem 3.3 $f(G_0, \gamma) = \begin{cases} \lfloor \frac{m}{3} \rfloor + 1 & \text{if } m \equiv 0 \pmod 3 \\ \lfloor \frac{m}{3} \rfloor + 3 & \text{otherwise} \end{cases}$ *for $r \equiv 0 \pmod 4$.*

Proof If $m \equiv 0 \pmod 3$ the set $F = \{v_{21}, v_{2(r+4)}, v_{5(r+4)}, v_{8(r+4)}, ..., v_{m-1(r+4)}\}$ determine the unique minimum dominating set for G_0 then $f(G_0, \gamma) \leq \lfloor m/3 \rfloor + 1$. But $\gamma(G_i) = 2\lfloor r/4 \rfloor$ for $\lfloor m/3 \rfloor$ of G_is. Hence $f(G_0, \gamma) = \lfloor m/3 \rfloor + 1$. The proof of the case $m \equiv 1$ or $2 \pmod 3$ is similar to the previous case. □

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