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On the Pseudo Smarandache function and its two conjectures

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Abstract For any positive integer n, the famous Pseudo Smarandache function $Z(n)$ is defined as the smallest integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is, $Z(n) = \min\left\{m:\ n|\frac{m(m+1)}{2},\ m \in N\right\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary method to study the properties of the Pseudo Smarandache function $Z(n)$, and solve two conjectures posed by Kenichiro Kashihara in reference [2].

Keywords Pseudo Smarandache function, conjecture, unbounded.

§1. Introduction and Results

For any positive integer n, the famous Pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is,

$$Z(n) = \min\left\{m:\ n|\frac{m(m+1)}{2},\ m \in N\right\},$$

where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are:

$$Z(1) = 1,\ Z(2) = 3,\ Z(3) = 2,\ Z(4) = 7,\ Z(5) = 4,\ Z(6) = 3,\ Z(7) = 6,\ Z(8) = 15,$$
$$Z(9) = 8, Z(10) = 4,\ Z(11) = 10,\ Z(12) = 8,\ Z(13) = 12,\ Z(14) = 7,\ Z(15) = 5,$$
$$Z(16) = 31,\ Z(17) = 16,\ Z(18) = 8,\ Z(19) = 18,\ Z(20) = 15,\ \cdots\cdots.$$

This function was introduced by David Gorski in reference [3], where he studied the elementary properties of $Z(n)$, and obtained a series interesting results. For example, he proved that if $p \geq 2$ is a prime, then $Z(p) = p - 1$; If $n = 2^k$, then $Z(n) = 2^{k+1} - 1$. The other contents related to the Pseudo Smarandache function can also be found in references [2], [4] and [5]. Especially in reference [2], Kenichiro Kashihara posed two problems as follows: Are the following values bounded or unbounded?

A) $|Z(n + 1) - Z(n)|$,

B) $\dfrac{Z(n + 1)}{Z(n)}$.

About these two problems, it seems that none had studied it, at least we have not seen related papers before. In this paper, we use the elementary method to study these two problems, and prove that they are unbounded. That is, we shall prove the following conclusion:

Theorem. For any positive number M large enough, there are infinitely positive integers n, such that

$$\frac{Z(n+1)}{Z(n)} > M \text{ and } |Z(n+1) - Z(n)| > M.$$

From this theorem, we know that $|Z(n+1) - Z(n)|$ and $\frac{Z(n+1)}{Z(n)}$ are unbounded. This solved two problems posed by Kenichiro Kashihara in reference [2].

§2. Proof of the theorem

In order to complete the proof of the theorem, we need the following important conclusion:

Lemma. Let k and h are any positive integers with $(h,\ k) = 1$, then there are infinitely many primes in the arithmetic progression $nk + h$, where $n = 0,\ 1,\ 2,\ 3,\ \cdots$.

Proof. This is the famous Dirichlet's Theorem, see reference [6].

Now we use this Lemma to complete the proof of our Theorem. In fact for any positive number M, we take positive integer m such that $2^m > M$. Note that $\left(2^{2m+1},\ 2^m + 1\right) = 1$, so from Dirichlet's Theorem we can easily deduce that there are infinitely many primes in the arithmetic progression:

$$2^{2m+1}k + 2^m + 1, \quad \text{where} \quad k = 0,\ 1,\ 2,\ \cdots.$$

Therefore, there must exist a positive integer k_0 such that $2^{2m+1}k_0 + 2^m + 1 = P$ be a prime. For this prime P, from the definition and properties of $Z(n)$ we can deduce that

$$Z(P) = P - 1 = 2^{2m+1}k_0 + 2^m,$$
$$Z(P-1) = Z(2^{2m+1}k_0 + 2^m) = Z(2^m(2^{m+1}k_0 + 1)).$$

Since

$$\sum_{i=1}^{2^{m+1}k_0} i = \frac{2^{m+1}k_0(2^{m+1}k_0 + 1)}{2}$$

and $2^m(2^{m+1}k_0 + 2^m)$ evenly divides $\sum_{i=1}^{2^{m+1}k_0} i$, so we have

$$Z(P-1) \leq 2^{m+1}k_0.$$

Thus

$$\frac{Z(P)}{Z(P-1)} \geqslant \frac{2^{2m+1}k_0 + 2^m}{2^{m+1}k_0} > 2^m > M.$$

So $\frac{Z(P)}{Z(P-1)}$ is unbounded.

Similarly, we have

$$\begin{aligned} |Z(P) - Z(P-1)| &\geq |Z(P)| - |Z(P-1)| \\ &\geq 2^{2m+1}k_0 + 2^m - 2^{m+1}k_0 \\ &= 2^{m+1}k_0(2^m - 1) + 2^m > 2^m > M. \end{aligned}$$

So $|Z(P) - Z(P-1)|$ is also unbounded.

Since there are infinitely positive integers m, such that $2^m > M$, so there are infinitely positive integers n, such that $|Z(n+1) - Z(n)|$ and $\dfrac{Z(n+1)}{Z(n)}$ are unbounded.

This completes the proof of the theorem.

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