ON Some Characterization of Smarandache Lattice with Pseudo complement

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Abstract:

In this paper we have introduced smarandache - 2 - Algebraic structure of lattice namely smarandache lattice. A smarandache 2- algebraic structure on a set N means a weak algebraic structure A_o on N such that there exists a proper subset M of N which is embedded with a stronger algebraic structure A_1, Stronger algebraic structure means that it is satisfying more axioms, by proper subset one understands a subset different from the empty set, from the unit element if any, and from the whole set. we define smarandache lattice and obtain some of its characterization through Pseudo complemented .For basic concept we refer to PadilaRaul[4].

Keywords: Lattice, Boolean Algebra, Smarandache lattice, Pseudo complemented lattice.

1. Introduction:

In order that New notions are introduced in algebra to better study the congruence in number theory by Florentin smarandache [1] .By <proper subset>of a set A we consider a set P included in A, and different from A, different from the empty set, and from the unit element in A – if any they rank the algebraic structures using an order relationship:

They say that the algebraic structures $S_1 \ll S_2$ if :both are defined on the same set;: all S_1 laws are also S_2 laws ; all axioms of an S_1 law are accomplished by the corresponding S_2 law;S_2law accomplish strictly more axioms than S_1 laws, or S_2 laws has more laws than S_1.

For example : Semi group $\ll$ monoid $\ll$ group $\ll$ ring $\ll$ field, or Semi group $\ll$ commutative semi group, ring $\ll$ unitary ring, etc.they define a General special structure to be a structure SM on a set A, different from a structure SN, such that a proper subset of A is an SN structure, where SM $\ll$ SN.

2. Preliminaries

Definition 2.1: Let S be a lattice with 0. Let $x \in S$ x*is a Pseudo complementet of x iff $x^* \in S$ and $x \wedge x^* = 0$ and for every $y \in S$: if $x \wedge y = 0$ then $y \leq x^*$.

Definition 2.2: S is pseudo complemented iff every element of S has a pseudo complement. Let S be a pseudo complemented lattice .$N_S = \{x^*: x \in S\}$,the set of pseudo complements in S $N_S = \{N_S \ \leq_N, \neg N, , 0_N, 1_N, \wedge_N, \vee_N\}$ where:

(i). $\leq_{N,}$ is defined by :for every $x, y \in Ns$: $x \leq_N y$ iff $x \leq_S b$

(ii). $\neg$ N is defined by :

for every $x \in N_S$: $\neg N(x) = x^*$

(iii). $\wedge_N$ is defined by:

for every $x, y \in N_S$ $x \wedge_N y = x \wedge_S y$

(iv). $\vee_N$ is defined by :

for every $x, y \in N_S$: $x \vee_N y = (x^* \wedge_S y^*)^*$

(vi). $1_N = 0_S^*$, $0_N = 0_S$

Definition 2.3: If S is a distributive lattice with 0,I_S is a complete Pseudo Complemented lattice. Let S be a lattice with 0. NI_S, the set of normal ideals in S, is given $NI_S = \{I^* \in I_S : I \in I_S\}$.

Definition 2.4: A pseudo complemented distributive lattice S is called a stone lattice if, for all $a \in S$, it satisfies the property $a \vee a^{**} = 1$.

Definition 2.5: Let S be a pseudo complemented distributive lattice. Then for any filter F of S, define the set $\delta(F)$ as follows $\delta(F) = \{a^* \in S / a^* \in F\}$.

Definition 2.6: Let S be a pseudo complemented distributive lattice, An ideal I of S is called a δ -ideal if $I = \delta(F)$ for some filter F of S.

Now we have introduced a definition by [4]:

Definition:

A lattice S is said to be a Smarandache lattice. If there exist a proper subset L of S, which is a Boolean Algebra with respect to the same induced operations of S.

3. Characterizations

Theorem 3.1: Let (S, $\wedge$, $\vee$,) be a lattice. If there exist a proper subset N_S of S , where $N_S = \{x^*: x \in S\}$is the set of all Pseudo complemented lattice in S .Then S is a smarandache lattice.

Proof: By hypothesis , let (S, $\wedge$, $\vee$,) be a lattice and whose proper subset $N_S = \{x^*: x \in S\}$the set of all pseudo complemented lattice in S.

It is enough to prove that N_S is a Boolean Algebra.

For ,

(i) for every x, y $\in N_S$,

$x \wedge_N y \in N_S$ and $\wedge_N$ is meet under $\leq_N$.

if x, y $\in N_S$, then $x=x^{**}$ and $y=y^{**}$.

Since $x \wedge_S y \leq_S x$,by result $x \leq_N y$ then $y^* \leq_N x^*$,

$x^* \leq_S (x \wedge_S y)^*$, and, with by result $x \leq_N y$ then $y^* \leq_N x^*$,

$(x \wedge_S y)^{**} \leq_S x$. Similarly, $(x \wedge_S y)^{**} \leq_S y$.

Hence $(x \wedge_S y)^{**} \leq_S (x \wedge_S y)$.

By result, $x \leq_N x^{**}$, $(x \wedge_S y) \leq_S (x \wedge_S y)^{**}$,

Hence $(x \wedge_S y) \in N_S$, $(x \wedge_N y) \in N_S$.

If $a \in N_S$ and $a \leq_N x$ and $a \leq_N y$,

Then $a \leq_S x$ and $a \leq_S y$, $a \leq_S (x \wedge_S y)$,

Hence $a \leq_N (x \wedge_N y)$.So indeed $\wedge_N$ is meet in $\leq_N$.

(ii) For every $x,y \in N_S : x \vee_N y \in N_S$ and $\vee_N$ is join under $\leq_N$.

Let $x,y \in N_S$. Then $x^*, y^* \in N_S$. Then by (i), $(x^* \wedge_S y^*) \in N_S$.

Hence $(x^* \wedge_S y^*)^* \in N_S$ and hence $(x \vee_N y) \in N_S$.

$(x^* \wedge_S y^*) \leq_S x^*$, hence, by result $x \leq_N x^{**}$

$X^{**} \leq_S (x^* \wedge_S y^*)^*$,

By result $N_S = \{x \in S;\ x = x^{**}\}$, $x \leq_S (x^* \wedge_S y^*)^*$.

Similarly, $y \leq_S (x^* \wedge_S y^*)^*$.

If $a \in N_S$ and $x \leq_N a$ and $y \leq_N a$, then $x \leq_S a$ and $y \leq_S a$,then by result if $x \leq_N y$

Then $y^* \leq_N x^*$, $a^* \leq_S x^*$ and $a^* \leq_S y^*$, hence $a^* \leq_S (x^* \wedge_S y^*)$

Hence ,by result if $x \leq_N y$ then $y^* \leq_N x^*$,

$(x^* \wedge_S y^*) \leq_S a^{**}$ hence, by result $N_S = \{x \in S;\ x = x^{**}\}$,

$(x^* \wedge_S y^*)^* \leq_S a$, hence $x \vee_N y \leq_N a$ so, indeed $\vee_N$ is join in $\leq_N$.

(iii) $0_N, 1_N \in N_S$ and 0_N and 1_N are the bounds of N_S.

Obviously $1_N \in N_S$, since $1_N = 0_S^*$ since for every $a \in N_S$, $a \wedge_S 0_S = 0_S$,

For every $a \in N_S$, $a \leq_S 0_S^*$,hence $a \leq_N 1_N$.

$0_S^*, 0_S^{**} \in N_S$. Hence $0_S^* \wedge_S 0_S^{**} \in N_S$. But of course, $0_S^* \wedge_S 0_S^{**} = 0_S$.

Hence $0_S \in N_S$, $0_N \in N_S$,Obviously, for every $a \in N_S$; $0_S \leq_S a$.

Hence for every $a \in N_S : 0_N \leq_N a$ .So N_S is bounded lattice.

(iv) For every $a \in N_S : \neg_N(a) \in N_S$ and for every $a \in N_S : a \wedge_N \neg_N(a) = 0_N$ and
For every $a \in N_S : a \vee_N \neg_N(a) = 1_N$.
Let $a \in N_S$, Obviously $\neg_N(a) \in N_S$.
$a \vee_N \neg_N(a) = a \vee_N a^* = ((a^* \wedge_S b^{**}))^* = (a^* \wedge_S a)^* = 0_S^* = 1_N$
$a \wedge_N \neg_N(a) = a \wedge_S a^* = 0_S = 0_N$.

So N_S is a bounded complemented lattice.

(v) Distributive.

Let $x,y,z \in N_S$. Since $x \leq_N (x \vee_N (y \wedge_N z))$,

$(x \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$

Also $(y \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$

Obviously, if $a \leq_N b$, then $a \wedge_N b^* = 0_N$, Since $b \wedge b^* = 0_N$,

Hence, $(x \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))^* = 0_N$ and

$(y \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))^* = 0_N$,

$x \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))^*) = 0_N$,

$y \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))^*) = 0_N$.

By definition of Pseudocomplement:

$z \wedge_N (x \vee_N (y \wedge_N z))^* \leq_N x^*$,

$z \wedge_N (x \vee_N (y \wedge_N z))^* \leq_N y^*$

Hence $z \wedge_N (x \vee_N (y \wedge_N z))^* \leq_N x^* \wedge_N y^*$

Once again, If $a \leq_N b$, then $a \wedge_N b^* = 0_N$,

Hence, $z \wedge_N (x \vee_N (y \wedge_N z))^* \wedge (x^* \wedge_N y^*)^* = 0_N$

$z \wedge_N (x^* \wedge_N y^*)^* \leq_N (x \vee_N (y \wedge_N z))^{**}$ Now, by definition of $\vee_N$:

$z \wedge_N (x^* \vee_N y^*)^* = z \wedge_N (x \vee_N y)$ And by $N_s = \{ x \in s, : x = x^{**}\}$:

$(x \vee_N (y \wedge_N z))^{**} = x \vee_N (y \wedge_N z)$, Hence : $z \wedge_N (x \vee_N y) \leq_N x \vee_N (y \wedge_N z)$.

Hence, indeed N_s is a Boolean Algebra.

Therefore by definition , S is a smarandache lattice.

Theorem 3.2: Let S be a distributive lattice with 0. If there exist a proper subset I_S of S , where I_S is the set of all ideals in S. Then S is a Smarandach**e** lattice.

Proof: By hypothesis ,let S be a distributive lattice with 0 and whose proper subset

I_S is the set of all ideals in S.

We claim that I_S is a Boolean algebra.

Let $I \in I_S$.Take $I^*=\{y \in S$:for every $i \in I: y \wedge i=0\}, I^* \in I_S$

Namely if $a \in I^*$ then for every $i \in I: a \wedge i=0$,

Let $b \leq a$, Then, obviously, for every $i \in I: b \wedge i=0$ hence $b \in I^*$.

If $a,b \in I^*$ then for every $i \in I: a \wedge i=0$,and for every $i \in I: b \wedge i=0$,

Hence for every $i \in I:(a \wedge i) \vee (b \wedge i)=0$.

With distributive, for every $i \in I$: $i \wedge (a \vee b)=0$,hence $a \vee b \in I^*$.

Hence $I^* \in I_S$,.$I \cap I^*=I \cap \{y \in S$:for every $i \in I: y \wedge i=0\}=\{0\}$.

Let $I \cap J=\{0\}$,let $j \in J$ Suppose that for some $i \in I: i \wedge j \neq 0$.

Then $i \wedge j \in I \cap J$, Since I and j are ideals, hence$I \cap J \neq \{0\}$.

Hence forevery $i \in I: j \wedge i=0$.,and hence $j \subseteq I^*$.

Consequently, I^* is a pseudo complement of I and I_S is a pseudo complemented .
In Theorem 3.1we have proved that pseudo complemented form a Boolean algebra .

Therefore I_S is a Boolean algebra .

Then by definition , S is a Smarandache lattice.

Theorem 3.3: Let S be a distributive lattice with 0. If there exist a proper subset NI_s of S , where $NI_s=\{I^* \in I_s, I \in I_s\}$ is the set of normal ideals in S. Then S is a smarandache lattice.

Proof: By hypothesis ,let S be a distributive lattice with 0 and whose proper subset $NI_s=\{I^* \in I_s$, $I \in I_s\}$ is the set of normal ideals in S.

We claim that NI_S is Boolean Algebra .

Since $NI_s=\{I^* \in I_s : I \in I_s\}$ is the set of normal ideals in S.

Alternatively $NI_s=\{I \in I_s : I= I^{**}\}$.

Thus NI_s is the set of all Pseudo complemented lattice in I_s .

In Theorem 3.1we have proved that pseudo complemented form a Boolean algebra .

Therefore NI_S is a Boolean algebra .

Hence by definition , S is a smarandache lattice .

Theorem 3.4: Let S be a lattice. If there exist a Pseudo complemented distributive lattice L, X*(L) is a sub lattice of the lattice $I^{\delta}(L)$ of all δ–ideals of L,which is the proper subset of S . Then S is a Smarandache lattice.

Proof: By hypothesis ,let S be a lattice and there exist a Pseudo complemented distributive lattice L, X*(L) is a sub lattice of the lattice $I^{\delta}(L)$ of all δ–ideals of L,which is the proper subset of S .

Let $(a^*],(b^*] \in X^*(L)$,f or some $a,b \in$ **L.** Then clearly $(a^*] \cap (b^*] \in X^*(L)$.

Again, $(a^*] \cup (b^*] = \delta([a)) \cup \delta([b))= \delta [(a) \cup ([b))= \delta ([a \cap b))=((a \cap b)^*] \in X^*(L)$.

Hence X*(L) is a sub lattice of $I^{\delta}(L)$ and hence a distributive lattice.

Clearly (0**] and (0*] are the least and greatest elements of X*(L).

Now for any $a \in$ **L**, $(a^*] \cap (a^{**}] = (0]$ and

$(a^*] \cup (b^{**}] = \delta([a)) \cup \delta([a^*)) = \delta([a)) \cup ([a^*)) = \delta([a \cap a^*))= \delta([0)) = \delta(L) = L$.

Hence (a**] is the complement of (a*] in X*(L). Therefore $\{X^*(L), \cup, \cap\}$is a bounded distributive lattice in which every element is complemented.

Thus X*(L) is a Boolean Algebra.

By definition, S is a Smarandache lattice.

Theorem 3.5 Let S be a lattice**,** L be a pseudo complemented distributive lattice.

If S is a Smarandache lattice.Then the following conditions are equivalent:

(a). L is a Boolean algebra.

(b). every element of L is closed,

(c). every principal ideal is a δ -ideal,

(d). for any ideal I, a $\in$ I implies a** $\in$ I,

(e). for any proper ideal I, I $\cap$ D(L) = ϕ ,

(f). for any prime ideal P, P $\cap$ D(L) = ϕ ,

(g). every prime ideal is a minimal prime ideal,

(h). every prime ideal is a δ –ideal,

(i). for any a,b $\in$ S, a* = b* implies a = b,

(j). D (L) is a singleton set.

Proof: Since S is a Smarandache lattice. Then by definition, there exist a proper subset L of S such that which is a Boolean algebra.

Therefore L is a Boolean algebra.

Now to prove

(a) $\Rightarrow$ **(b):** Then clearly L has a unique dense element, precisely the greatest element.

Let a $\in$ S. Then $a^* \wedge a = 0 = a^* \wedge a^{**}$. Also $a^* \vee a$, $a^* \vee a^{**} \in$ D (L).

Hence $a^* \vee a = a^* \vee a^{**}$. By the cancellation property of S, we get a =a**.

Therefore every element of L is closed.

(b)$\Rightarrow$(c): Let I be a principal ideal of L. Then I = (a] for some a $\in$ L .Then by condition (b),

a = a**. Now, (a] = (a**] = δ ([a*)). Therefore (a)] is a δ-ideal.

(c)$\Rightarrow$(d): I be a proper ideal of L. Let a $\in$ I. Then (a] = δ (F) for some filter F of L. Hence

We get a***=a* $\in$ F. Therefore a**$\in$ δ(F) = (a]$\subseteq$I.

(d)$\Rightarrow$(e): Let I be a proper ideal of L. Suppose a $\in$I $\cap$D (S). Then a** $\in$ S and a*=0.

Therefore 1=0* = a** $\in$ L, which is a contradiction.

(e)$\Rightarrow$(f): Let I be a proper ideal of S,I$\cap$D(L) =ϕ ,then P be a prime ideal of L ,P$\cap$D(L) = ϕ.

(f)$\Rightarrow$(g): Let P be a prime ideal of S such that P$\cap$D (L) =ϕ. Let a$\in$ P. Then clearly a$\wedge$ a*= 0

and a $\vee$a* $\in$D (L). Hence a $\vee$a* $\notin$ p Thus a*$\notin$ p Therefore P is a minimal prime ideal of L.

(g)$\Rightarrow$(h): Let P be a minimal prime ideal of L. Then clearly L-P is a filter of L. Let a $\in$P.

Since P is minimal, there exists b $\notin$ P such that a $\wedge$ b = 0. Hence a*$\wedge$ b = b.

a* $\notin$ P. Thus a^* $\in$ (S-P) which yields a$\in$ δ (L-P). Conversely, let a $\in$ δ(L-P). Then we get

a* $\notin$ P. Hence we have a $\in$P. Thus P=δ (L-P). And therefore P is δ - ideal of L.

(h)$\Rightarrow$(i): Assume that every prime ideal of L is a δ – ideal. Let a, b $\in$ L be such that a*=b*.

Suppose a$\neq$b. Then there exists a prime ideal P of S such that a$\in$P and b$\notin$ P. By

Hypothesis, P is a δ - ideal of L. Hence P = δ(F) for some filter F of L. Since a $\in$P =δ(F),

We get b* = a*$\in$ F. Hence b$\in$ δ (F) = P, which is a contradiction. Therefore a=b.

(i)$\Rightarrow$(j): Suppose x, y be two elements of D (L). Then x * = 0 = y *. Hence x = y.

Therefore D (L) is a singleton set.

(j)$\Rightarrow$(a): Assume that D (L) = {d} is singleton set. Let a$\in$ L. We have always a$\vee$ a* D (L).

Therefore a $\wedge$ a* = 0 and a $\vee$ a*=d. This true for all a$\in$ L. Also 0$\leq$a$\leq$a$\vee$a*=d.

Hence the above conditions are equivalent.

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