On Smarandache's Sphere

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Through one of the intersecting points of two circles we draw a line that intersects a second time the circles in the points M_1 and M_2 respectively. Then the geometric locus of the point M which divides the segment M_1M_2 in a ratio k (i.e. $M_1M = k \cdot MM_2$) is the circle of center O (where O is the point that divides the segment of line that connects the two circle centers O_1 and respectively O_2 into the ratio k, i.e. $O_1O = k \cdot OO_2$) and radius OA, without the points A and B.

Proof

Let $O_1E \perp M_1M_2$ and $O_2F \perp M_1M_2$. Let $O \in O_1O_2$ such that $O_1O = k \cdot OO_2$ and $M \in M_1M_2$, where $M_1M = k \cdot MM_2$.

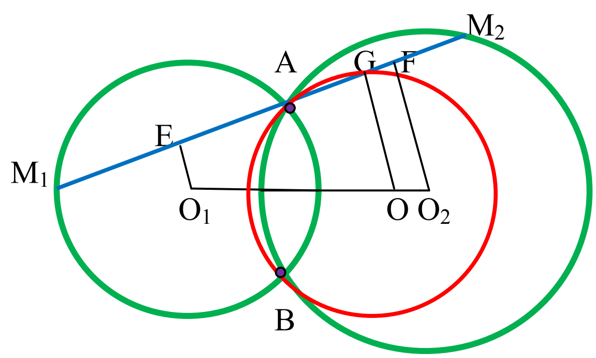

Fig. 1.

We construct $OG \perp M_1M_2$ and we make the notations: $M_1E \equiv EA = x$ and $AF \equiv FM_2 = y$.

Then, $AG \equiv GM$, because

$$AG = EG - EA = \frac{k}{k+1}(x+y) - x = \frac{-x+ky}{k+1}$$

and

$$GM = M_1M - M_1A - AG = \frac{k}{k+1}(2x+2y) - 2x - \frac{-x+ky}{k+1} = \frac{-x+ky}{k+1}.$$

Therefore we also have $OM \equiv OA$.

The geometric locus is a circle of center O and radius OA, without the points A and B (the red circle in Fig. 1).

Conversely.

If $M \in (GO, OA) \setminus \{A, B\}$, the line AM intersects the two circles in M_1 and M_2 respectively.

We consider the projections of the points O_1, O_2, O on the line M_1M_2 in E, F, G respectively. Because $O_1O = k \cdot OO_2$ it results that $EG = k \cdot GF$.

Making the notations: $M_1E \equiv EA = x$ and $AF \equiv FM_2 = y$ we obtain that

$$M_1M = M_1A + AM = M_1A + 2AG = 2x + 2(EG - EA) =$$
$$= \left[2x + 2\frac{k}{k+1}(x+y) - x\right] = \frac{k}{k+1}(2x+2y) = \frac{k}{k+1}M_1M.$$

For $k = 2$ we find the Problem IV from [1].

Generalizations.

1) The same problem if instead of two circles one considers two ellipses, or one ellipse and one circle.
2) The same problem in *3D,* considering instead of two circles two spheres (their surfaces) whose intersection is a circle $\mathcal{C}$. Drawing a line passing through the circumference of $\mathcal{C}$, it will intersect the two spherical surfaces in other two points M_1 and respectively M_2.
   Conjecture: The geometric locus of the point M which divides the segment M_1M_2 in a ratio *k* (i.e. $M_1M = k \cdot MM_2$) includes the spherical surface of center *O* (where *O* is the point that divides the segment of line that connects the two sphere centers O_1 and respectively O_2 into the ratio *k*, i.e. $O_1O = k \cdot OO_2$) and radius *OA*, without the intersection circle $\mathcal{C}$ [*Smarandache's Sphere*].
   A partial proof is this: if the line M_1M_2 which intersect the two spheres is the same plane as the line O_1O_2 then the *3D* problem is reduce to a *2D* problem and the locus is a circle of radius *OA* and center *O* defined as in the original problem, where the point *A* belongs to the circumference of $\mathcal{C}$ (except two points). If we consider all such cases (infinitely many actually), we get a sphere of radius *OA* (from which we exclude the intersection circle $\mathcal{C}$) and centered in *O (A* can be any point on the circumference of intersection circle $\mathcal{C}$*).*
   The locus has to be investigated for the case when M_1M_2 and O_1O_2 are in different planes.
3) What about if instead of two spheres we have two ellipsoids, or a sphere and an ellipsoid?

References:

[1] The Admission Test at the Polytechnic Institute, *Problem IV*, 1987, Romania.
[2] Florentin Smarandache, *Proposed Problems of Mathematics (Vol. II),* University of Kishinev Press, Kishinev, Problem 58, pp. 38-39, 1997.

[3] F. Smarandache, *Nine Solved and Nine Open Problems in Elementary Geometry*, in arXiv.org.