Unique Einstein Gravity from Feynman's Lorentz Condition

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Abstract

The Principle of Equivalence is a central concept of Einstein's theory of gravitation, which makes generally covariant presentation of that theory extremely convenient for working out and communicating its dynamical detail. Weinberg has pointed out, however, that part of the base physics content of the theory, e.g., its overall stress-energy and conserved four-momentum, is inaccessible to formulations which maintain strict general covariance. Feynman, whose lectures on gravitation initially pursue an order-by-order construction of the theory, as a by-product arrived at a revelatory insight into the dynamical character of the gravitational field that reaches one step beyond Weinberg's not yet theory-stipulating separation of the Einstein tensor into its physically dissimilar linear and nonlinear parts. While Einstein's gauge-imposition of a local conservation principle is a common theme in physical theory, permitting it to spill over into outright solution nonuniqueness clearly isn't acceptable classical physics. Thus stipulation of Feynman's gravitational Lorentz condition, which pins down the dynamical character of the gravitational field, is fundamental to physically sensible application of Einstein's gravity theory.

Introduction

Newton postulated that any reference frame which is at rest or in uniform motion relative to "absolute space" is an inertial one in which there exist no inertial forces such as centrifugal force. Mach pointed out that it seemed less metaphysical to suppose that inertial forces such as centrifugal force arise from accelerations relative to actually existing massive bodies, such as the sun, stars and nebulae, rather than to accelerations relative to "absolute space".

Einstein's Principle of Equivalence postulates the universal existence of localized strictly inertial reference frames, in which *not only* inertial forces such as centrifugal force are absent, but all *gravitational* force *is absent as well*. Einstein's postulate doesn't envision these localized inertial reference frames as being at rest or in uniform motion relative to Newton's "absolute space", but rather as being "in free fall" in whatever the local mean gravitational field happens to be. Therefore Einstein's postulated local inertial frames of reference can certainly be accelerating relative to large masses in their vicinity, which contradicts Mach's supposition that such acceleration is the cause of inertial forces. Weinberg [1] points out that the available evidence strongly favors Einstein's postulate over Mach's supposition; for example, the experiment of Dicke [2] found, with high accuracy, no *local* force due to the sun's gravitational field, in which the Earth is falling freely, notwithstanding the Earth's consequent acceleration relative to the sun (although there *do* exist less local *tidal* effects due to the *gradient* of the sun's gravitational field).

Einstein's gravitational theory is strongly aligned to his Principle of Equivalence, which can *seem* to produce departures from normal dynamical reasoning that is consonant with Einstein's 1905 Lorentz-covariant relativity. For example, in Einstein's local inertial frames gravitational *force* is absent, so *all* of the stress-energy present in such frames can be attributed to "matter", which implies that the "matter" stress-energy tensor must have vanishing divergence in such frames. The mathematical instrument used in Einstein's gravity theory to select differential properties of theoretical-physics entities *in local inertial frames* is the *generally covariant* partial derivative. Therefore the *generally covariant* divergence of the "matter" stress-energy tensor must vanish.

The fundamental gravitational field equation of Einstein's gravity theory states a nonlinear *second-order differential* relation of the gravitational potential (aka the "metric" tensor) to the "matter" stress-energy tensor in local inertial frames—although gravitational *force* is *absent* in such frames, that *isn't* the case for either the gravitational *potential* or the gravitational *force gradient* (aka the "curvature") which occurs in Einstein's gravitational field equation in conjunction with the "matter" stress-energy tensor, and is a nonlinear second-order differential transformation of the gravitational potential. In *highly abbreviated form* Einstein's gravitational field equation is written,

$$G_{\mu\nu} = -((8\pi G)/c^4)T_{\mu\nu}, \tag{1a}$$

where $T_{\mu\nu}$ on its right-hand side is the symmetric "matter" stress-energy tensor, and the symmetric tensor $G_{\mu\nu}$ on its left-hand side is Einstein's special form of the contracted Riemann curvature tensor that is

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nonlinearly made up of zeroth, first and second partial derivatives of the symmetric gravitational potential (i.e., "metric") tensor $g_{\mu\nu}$. The theoretical physics thrust of this Einstein gravitational field equation is, of course, the mathematically detailed way in which the "matter" stress-energy tensor $T_{\mu\nu}$ acts as the *source* of the gravitational potential (i.e., of the "metric" tensor) $g_{\mu\nu}$ *in the context of local inertial frames* where the gravitational *force* vanishes.

We have pointed out that in such frames the *only* stress-energy is that of the "matter", which in turn implies the vanishing of the divergence of $T_{\mu\nu}$ *in those local inertial frames*, and that this is mathematically expressed as the vanishing of the *generally covariant* divergence of $T_{\mu\nu}$, namely,

$$T_{\mu\nu;}{}^{\nu} = 0, \tag{1b}$$

where the semicolon and index are used to denote the quite complicated *generally covariant* partial derivative, which is a construct that includes nonlinear combinations of the zeroth and first partial derivatives of the gravitational potential (i.e., of the "metric") because such a generally covariant partial derivative in fact accounts *in more general frames* for the gravitational *force* which happens to be *absent* in local inertial frames.

Einstein specifically *constructed* his form $G_{\mu\nu}$ of the contracted Riemann curvature tensor in such a way that his gravitational field equation, namely Eq. (1a), *compels* the "matter" stress-energy tensor to have vanishing generally covariant divergence, i.e., he made Eq. (1a) self-inconsistent *unless* Eq. (1b) is satisfied. Einstein did this by contracting $G_{\mu\nu}$ from the Riemann curvature tensor in such a way that,

$$G_{\mu\nu;}{}^{\nu} = 0, \tag{1c}$$

is *identically* satisfied—he made use of a Riemann curvature-tensor identity known as the Bianchi identity to accomplish this [3].

It *does* seem desirable in principle to make such a fundamental "local conservation" property of a theory as Eq. (1b) a *consequence* of its *basic equation*, which in *this* instance is Einstein's gravitational field equation, Eq. (1a). Maneuvers along the lines of Einstein's *specifically designing* his $G_{\mu\nu}$ such that Eq. (1c) is an *identity* for the *express purpose* of *absorbing* a local conservation property into a theory's *basic equation* have indeed become *very widespread* in theoretical physics practice since Einstein's time, but *like all contrivances* such "gauge" imposition of a local conservation property on a theory's basic equation *comes with a built-in potential downside* which needs to be kept firmly in mind, as we point out in the next paragraph. The "poster child" of "gauge" imposition is of course electromagnetic theory, whose *basic* four-vector potential equation compels the vanishing of the divergence of the four-vector current density j^ν when it is written in the form,

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = j^\nu / c. \tag{2a}$$

The left-hand side of Eq. (2a) has been *specifically contrived* to have the property that its *normal* divergence vanishes *identically*, which *thereby* compels the *desired* vanishing of the *normal* divergence of the four-current density j^ν on its right-hand side as a *direct consequence* of this *particular* electromagnetic basic equation.

Doing this sort of "neat trickery", however, drops a fly into the ointment: insofar as a *part* of the basic equation is caused to be *identically satisfied*, a *redundancy* is introduced into that equation which *prevents* it from having a *unique* solution; i.e., *that basic equation can no longer unequivocally supply all of the information that it was envisioned to supply when it was written down.*

In Einstein's gravity theory the tendency has been to "just live with" such a state of affairs, citing the fact that pure mathematicians who practice the closely related discipline of Riemann differential geometry do exactly that by design. That mathematical discipline specifically *limits its interest* to the "intrinsic" properties of the hypersurfaces that it studies, which are the *invariants* of general coordinate transformations. To obtain *those*, it isn't *necessary* for the basic equation to possess *unique* solutions; *any* solution, however nonunique, will do because coordinate-transformation *invariants* have *the same value* for *all solutions*.

Is it reasonable for the gravitational branch of *classical theoretical physics* to likewise *limit its predictive goals* concerning the results of envisioned measurements? Einstein inveighed to the end of his life against the "incompleteness" of quantum theory, obstinately deaf to the question of how the implications of particle-wave duality could possibly otherwise be theoretically dealt with. Yet here for a branch of *manifestly classical physics*, namely gravitational theory, "living with" basic equations that fail to provide *unique* measurement-predictive solutions is bafflingly regarded with *equanimity* by many authorities!

In the starkest contrast, *no one* advocates for a single moment "just living with" the solution nonuniqueness of the basic electromagnetic four-vector potential theory manifested by Eq. (2a) above. To be sure, the most *dramatic* way to deal with Eq. (2a), namely its *complete abandonment* in favor of the electromagnetic *field equation system*,

$$F^{\mu\nu} = -F^{\nu\mu}, \quad \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu F^{\mu\nu} = j^\nu/c,$$

which does indeed possess unique solutions for $F^{\mu\nu}$ and which is inspired by the relations which the *potential* entity $(\partial^\mu A^\nu - \partial^\nu A^\mu)$ satisfies, *doesn't seem to have a gravitational-field-equation analog that springs from similar relations and that likewise has unique solutions*. However there *also* exists *a less dramatic and much more straightforward*, but still fully Lorentz-covariant alternate tack *for very satisfactorily resolving the solution nonuniqueness issue of the electromagnetic* Eq. (2a), namely *adjunct stipulation* of the Lorentz condition,

$$\partial_\mu A^\mu = 0, \tag{2b}$$

which *when inserted into* Eq. (2a) has the *additional* virtue of considerably *simplifying* it to just,

$$\partial_\mu \partial^\mu A^\nu = j^\nu/c, \tag{2c}$$

As a matter of fact, Eqs. (2b) and (2c) carry with them theoretical physics benefits which go *far beyond* merely *resolving* the solution ambiguity of Eq. (2a): *they bring the physical characteristics of the electromagnetic field into sharp focus*, revealing that its radiation propagates at speed c and has two possible polarizations which lie in a plane transverse to its propagation direction, and also that its nonradiative component is in essence that of a static scalar field along the coulombic lines of the equation $-\nabla^2 A^0 = j^0/c$—see R. P. Feynman's systematic Fourier-transformation dissection of Eqs. (2b) and (2c) [4].

Indeed, Feynman [5] (and also W. E. Thirring [6]) have done the *analogous adjunct stipulation* of the "Lorentz condition" *for gravitation*, which in the course of likewise removing gravitation's solution ambiguity *simplifies* that bulky theory *even more drastically* than what we have just seen above in the case of electromagnetism, and it similarly brings *the physical characteristics* of the Lorentz pure-tensor dynamical gravitational potential $g_{\mu\nu}$ into sharp focus.

We note, however, that there is *no way* to successfully apply an adjunct *linear* "Lorentz condition" stipulation to the Einstein equation when it is presented in the customary "generally covariant" form of Eq. (1a); that fact is apparent from the *nonlinear* character of the Einstein tensor $G_{\mu\nu}$ which occurs on its left-hand side. Indeed, the Eq. (1a) form of the Einstein equation is unsuited to any manner of theoretical physics contemplation *which isn't completely tied to local inertial frames*: the "matter" stress-energy tensor on the right-hand side of Eq. (1a) *ignores* the stress-energy contribution associated with the gravitational field *by itself*, which stress-energy contribution is in turn *indiscriminately combined* with a key descriptor of the gravitational potential's dynamics into the Einstein tensor found on the left-hand side of Eq. (1a).

In the next section we follow Weinberg's lead [7] in *separating* the Einstein tensor into its *linear* descriptor of the gravitational potential's dynamics and its *nonlinear* exclusively gravitational-field *contribution to the total stress-energy*, which is then *combined* with the "matter" contribution to the total stress-energy. The resulting total stress-energy is then readily seen to have vanishing *normal* divergence (as it logically *must* have), and it is *also* seen to be *the complete source* which feeds the linear descriptor of the gravitational potential's dynamics, as also logically *must* be the case.

But notwithstanding this satisfactory resolution of the stress-energy mysteries of the Einstein equation, it is easily verified to *remain* an ambiguously *under-determined* redundant equation system which has *four* of its ten field degrees of freedom up for grabs. However, with the *details* of the linear descriptor of the gravitational potential's dynamics *in hand*, adjunct stipulation of Feynman's *linear* "Lorentz condition" for gravitation *can* be carried out, and has, if anything, even *more* of a salutary effect on that ambiguous *gravitational* equation than the *electromagnetic* Lorentz condition of Eq. (2b) has on the ambiguous electromagnetic four-vector potential of Eq. (2a).

The Einstein equation as an unambiguous dynamical field theory

Following Weinberg [7] we extract from the Einstein tensor $G_{\mu\nu}$ on the left-hand side of the Einstein equation presented in Eq. (1a) the part $G^{(1)}_{\mu\nu}$ of that tensor which is *linear* in $(g_{\mu\nu} - \eta_{\mu\nu})$, where $\eta_{\mu\nu}$ is the flat-space Minkowski metric, and send that tensor's *nonlinear remainder* to join the "matter" stress-energy tensor $T_{\mu\nu}$ on the equation's right-hand side. The result of carrying those steps out is,

$$G^{(1)}_{\mu\nu} = -((8\pi G)/c^4)\tau_{\mu\nu}, \tag{3a}$$

where with the definition, $\partial^\kappa \stackrel{\text{def}}{=} \eta^{\kappa\lambda}\partial_\lambda$,

$$G^{(1)}_{\mu\nu} \stackrel{\text{def}}{=} \tfrac{1}{2}[\partial_\sigma\partial^\sigma g_{\mu\nu} - \partial_\nu\partial^\sigma g_{\mu\sigma} - \partial_\mu\partial^\sigma g_{\nu\sigma} + \partial_\mu\partial_\nu\eta^{\alpha\beta}g_{\alpha\beta} + \eta_{\mu\nu}(\partial^\sigma\partial^\tau g_{\sigma\tau} - \partial_\sigma\partial^\sigma\eta^{\alpha\beta}g_{\alpha\beta})], \tag{3b}$$

and,

$$\tau_{\mu\nu} \stackrel{\text{def}}{=} T_{\mu\nu} - (c^4/(8\pi G))(G^{(1)}_{\mu\nu} - G_{\mu\nu}). \tag{3c}$$

It is readily verified from Eq. (3b) that $G^{(1)}_{\mu\nu}$ is, like $G_{\mu\nu}$ itself, *symmetric* in its two indices $\mu\nu$. With some algebraic effort it can *also* be verified that,

$$\partial^\mu G^{(1)}_{\mu\nu} = 0, \tag{3d}$$

is an *identity*. From Eq. (3a) this implies that,

$$\partial^\mu \tau_{\mu\nu} = 0, \tag{3e}$$

must also hold identically.

Eqs. (3d) and (3e) show that $\tau_{\mu\nu}$, whose *normal* divergence vanishes, is the grand total stress-energy of the entire system, which it indeed ought to be, since is has been constructed by combining the "matter" stress-energy $T_{\mu\nu}$ with the nonlinear part of the Einstein tensor that contains the stress-energy due to the gravitational field by itself. This grand total stress-energy $\tau_{\mu\nu}$ is seen from Eq. (3a) to be the *source* that feeds the linearized Einstein field tensor $G^{(1)}_{\mu\nu}$, which is a key descriptor of the dynamics of $g_{\mu\nu}$.

Of course Eqs. (3d) and (3e) show that the Einstein equation reexpressed in the form of Eq. (3a) still suffers from *the very same solution nonuniqueness* that we saw afflicted it when it was written in the very much less transparent "generally covariant" form of Eq. (1a). But with the *linear* character of the key descriptor $G^{(1)}_{\mu\nu}$ of the dynamics of $g_{\mu\nu}$ now *explicitly displayed*, we are in a position to attempt to tackle that issue with a well-chosen Lorentz-covariant *linear* adjunct stipulation.

For that purpose we of course select Feynman's gravitational "Lorentz condition" [5],

$$\partial^\mu g_{\mu\nu} = \tfrac{1}{2}\partial_\nu(\eta^{\alpha\beta}g_{\alpha\beta}). \tag{4a}$$

Insertion of this gravitational "Lorentz condition" into the six-term linearized Einstein tensor $G^{(1)}_{\mu\nu}$ that is explicitly given by Eq. (3b) veritably *collapses* the latter into just *two* terms, so that in conjunction with the above Eq. (4a) gravitational "Lorentz condition" the Eq. (3a) Einstein equation becomes simply,

$$\tfrac{1}{2}\partial_\sigma\partial^\sigma(g_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}g_{\alpha\beta}) = -((8\pi G)/c^4)\tau_{\mu\nu}. \tag{4b}$$

One can readily evaluate the contraction of both sides of Eq. (4b) with the Minkowski metric tensor $\eta^{\mu\nu}$. Using that information enables one to reexpress the Eq. (4b) form of the Einstein equation as,

$$\tfrac{1}{2}\partial_\sigma\partial^\sigma g_{\mu\nu} = -((8\pi G)/c^4)(\tau_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\tau_{\alpha\beta}), \tag{4c}$$

which is the form preferred by Feynman.

Feynman [5] uses Fourier transformation to dissect the Eq. (4c) form of the Einstein equation in conjunction with its "Lorentz condition" of Eq. (4a). He shows the $g_{\mu\nu}$ potential which satisfies those two equations to be a Lorentz pure-tensor dynamical field whose radiative part propagates at the speed of light and has two polarizations in a plane transverse to its direction of propagation. Its nonradiative part astonishingly has a more complicated structure than one would expect given the *scalar* character of Newtonian gravitostatics. Electromagnetism has a scalar nonradiative part very similar to that of Newtonian gravitostatics, but the Einstein equation's nonradiative part is *irreducibly tensor* in character. However that prominent tensor part of the static gravitational field negligibly affects *non-relativistically* moving probe masses; there is a *dynamical* $(v/c)^2$ suppression of its *effect*. But it certainly comes into its own for the deflection of *light* by the sun, for which it *doubles* the refractive bending from what would be expected from a straightforward extension of Newtonian gravitation.

Feynman also briefly considers just such a straightforward extension of Newtonian gravitation, which would simplify the gravitational Lorentz condition of Eq. (4a) to merely $\partial^\mu g_{\mu\nu} = 0$ and would as well *remove* the term $((8\pi G)/c^4)(\frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\tau_{\alpha\beta})$ from the right-hand side of Eq. (4c). Feynman finds that such a theory *no longer* has Lorentz pure-tensor character; its radiative part manifests *three* polarizations, the third polarization arising from an admixture of *scalar* field. Of course this straightforward extension of Newtonian gravitation fails the test of deflection of light by the sun by a factor of two, and it significantly deviates from observations of planetary perihelion precession as well.

Feynman's gravitational "Lorentz condition" of Eq. (4a) can be rewritten in the form,

$$\eta^{\alpha\beta}\partial_\alpha g_{\beta\nu} = \tfrac{1}{2}\partial_\nu(\eta^{\alpha\beta}g_{\alpha\beta}), \tag{5a}$$

or,

$$\eta^{\alpha\beta}(\partial_\alpha g_{\beta\nu} - \tfrac{1}{2}\partial_\nu g_{\alpha\beta}) = 0, \tag{5b}$$

or as well,

$$\tfrac{1}{2}\eta^{\alpha\beta}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) = 0, \tag{5c}$$

which clearly can be expressed in terms of the affine connection as,

$$\eta^{\alpha\beta}g_{\nu\sigma}\Gamma^\sigma_{\alpha\beta} = 0. \tag{5d}$$

Eq. (5d) is a stipulation that is mathematically equivalent to the linear part of *the harmonic coordinate condition* [8], namely,

$$g^{\alpha\beta}\Gamma^\tau_{\alpha\beta} = 0, \tag{5e}$$

which itself has indeed often been utilized is gravity theory, albeit without a clear understanding of its physical importance.

The affine connection $\Gamma^\lambda_{\mu\nu}$ can certainly be interpreted as *the gravitational field* in light of the central role it plays in the gravitational-trajectory geodesic equation, and the harmonic coordinate condition of Eq. (5e) specifies the vanishing of one of its *contractions*. Interestingly, the *analogous* vanishing of the contraction of the *electromagnetic field* $F_{\mu\nu}$, namely,

$$g^{\alpha\beta}F_{\alpha\beta} = 0,$$

doesn't need to be specified explicitly because of the antisymmetric property $F_{\nu\mu} = -F_{\mu\nu}$ of the electromagnetic field. This subtle slight difference between electromagnetic and gravitational field theory had, before the work of Thirring and Feynman, been an impediment to the realization of classical gravitation as well-defined, fully deterministic classical physics.

Adherence to the harmonic coordinate condition, which is the completion of Feynman's linearized gravitational "Lorentz condition", *adds* to the geometric physics which follows from the Principle of Equivalence *the characterization of* $\Gamma^\lambda_{\mu\nu}$ *as a dynamical field* whose radiation propagates at the speed of light and has two transverse polarization states—this characterization is the crucial *physical* contribution which Feynman and Thirring made to gravitational theory.

The harmonic *physically proper* form of the static, spherically symmetric *Schwarzschild metric* is [9],

$$ds^2 = \left(\tfrac{1-(\rho_0/r)}{1+(\rho_0/r)}\right)(cdt)^2 - \left(\tfrac{1+(\rho_0/r)}{1-(\rho_0/r)}\right)(dr)^2 - (1+(\rho_0/r))^2r^2((d\theta)^2 + (\sin\theta d\phi)^2), \tag{6}$$

where $\rho_0 = (GM/c^2)$. This metric has the property that to first order in G it is isotropic; it only departs from isotropy because of the character of higher-order corrections.

The metric of Eq. (6) has a horizon at $r = \rho_0 = (GM/c^2)$ which is *not* physically attainable because static gravity theory with feedback implies that the effective mass M of a static spherically-symmetric energy distribution which can be inscribed in a sphere of radius r_s is unattainably bounded above by $((c^2r_s)/G)$ [10].

Christoph Schiller's Principle of Maximum Force [11] *also* implies the unattainable upper bound of $((c^2r_s)/G)$ on the effective mass M of a static spherically symmetric energy distribution which can be inscribed in a sphere of of radius r_s. If *two* such identical spheres of effective mass M just touch, the magnitude of the gravitational force between their centers is $((GM^2)/(4r_s^2))$, which must be strictly less than the Schiller's Maximum Force value of $(c^4/(4G))$. This immediately yields the unattainable upper bound of $((c^2r_s)/G)$ on such a static spherically symmetric object's effective mass M.

References

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