ON SMARANDACHE RINGS

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Abstract:

It is proved that a ring R in which for every $x \in$ R there exists a (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$ is always a Smarandache Ring. Two examples are provided for justification.

Key words: Ring**,** Smarandache Ring, Field, Partially Ordered Set, Idempotent elements.

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Introduction

In [14], it is stated that, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B $\subset$ A which is embedded with a stronger structure S. These types of structures occur in our every day's life.

The study of Smarandache Algebraic structures was initiated in the year 1998 by Raul Padilla following a paper written by Florentin Smarandache called "Special Algebraic Structures". Padilla treated the Smarandache Algebraic Structures mainly with associative binary operation.

In, [11], [12], [13], [14], W.B. Vasantha Kandasamy has succeeded in defining around 243 Smarandache concepts by creating the Smarandache analogue of the various ring theoretic concepts.

The Smarandache notions are an excellent means to study local properties in Rings. The definitions of two levels of Smarandache rings, namely, S-rings of level I and S-rings of level II are given. S-ring level I, which by default of notion, will be called S-ring.

In [3] a ring R in which for every $x \in$ R there exists a (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$ is introduced. In the literature such rings exist naturally, for instance, the rings Z_6 (modulo integers), Z_{10} (modulo integers), Boolean ring. In this paper we prove that "A ring R in which for every $x \in$ R there exists a (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$ is always a Smarandache ring. Two examples are provided for justification.

In section I we recall some definitions, examples and propositions pertaining to Smarandache Rings. In section 2 we prove our main theorem. In section 3, we give examples to justify our theorem. For basic definitions and concepts please refer [3].

Section – I

Definition 1.1: ([13]). A Smarandache ring (in short S-ring) is defined to be a ring A such that a proper subset of A is a field with respect to the operations induced. By a proper subset we understand a set included in A different from the empty set, from the unit element if any and from A.

Example 1.2: Let F[x] be a polynomial ring over a field F. F[x] is an S-ring.

Example 1.3: Let $Z_{12} = \{0,1,2, \ldots, 11\}$ be a ring . Z_{12} is an S-ring as $A = \{0,4,8\}$ is a field with 4 acting as the unit element.

It is interesting to note that we do not demand the unit of the ring to be the unit of the field.

Definition 1.4: Let R be a ring. R is said to be a Smarandache ring of level II (S-ring II) if R contains a proper subset A ($A \neq \phi$) such that

(1.4.1) A is an additive abelian group

(1.4.2) A is a semi group under multiplication

(1.4.3) For $a, b \in$ A, $a.b = 0$ if and only if $a = 0$ or $b = 0$.

Proposition 1.5: ([13]) Let R be an S – ring I, then R is an S – ring II.

Proposition 1.6: ([4]). Any finite domain is a division ring.

Section – 2

In this section we show that the ring R in which for every element $x \in$ R there exists *a* (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is a Smarandache ring. For completeness, we write some lemmas from [1].

In [3], it is well known that the ring R in which for every element $x \in$ R there exists *a* (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is commutative and $x^{n(x)-1}$ is an idempotent element of R, i.e, for every $x \in$ R,

$$(x^{n(x)-1})^2 = x^{n(x)-1}, \tag{i}$$

which implies that R has no nonzero nilpotent elements i.e., for every $x \in$ R and every natural number $k \geq 1$

$$x^k = 0 \text{ implies } x = 0 \tag{ii}$$

Lemma 2.1: Let R be a ring in which for every element $x \in$ R there exists *a* (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. The ring R is partially ordered by $\leq$ where for all elements x and y of R

$$x \leq y \text{ if and only if } xy = x^2 \tag{iii}$$

Proof: It is immediate that $\leq$ is reflexive as $xx = x^2$. Next, let $x \leq y$ and $y \leq x$ then $xy = x^2$ and $yx = y^2$. Now $(x^2 - xy) - (yx - y^2) = 0 \Rightarrow x^2 - xy - yx + y^2 = 0 \Rightarrow x^2 - xy - xy + y^2 = 0$ as R is commutative. This implies that $x^2 - 2xy + y^2 = 0 \Rightarrow (x - y)^2 = 0$. In view of (ii), we get $x - y = 0$ or $x = y$. Hence, $\leq$ is anti-symmetric. Finally, let $x \leq y$ and $y \leq z$, i.e., $xy = x^2$ and $yz = y^2$. Now, $x^2z = xyz = xy^2 = x^2y = x^3$. So $x^2z = x^3 \Rightarrow x^2z^2 = x^3z$ and $x^3z = x^4$. But, then $(x^2z^2 - x^3z) - (x^3z - x^4) = 0 \Rightarrow x^2z^2 - 2x^3z + x^4 = 0 \Rightarrow (xz - x^2)^2 = 0$. In view of (ii), we get $xz = x^2 \Rightarrow x \leq z$. Therefore, $\leq$ is transitive. Hence, $(R, \leq)$ is a partially ordered set.

Lemma 2.2: Let R be a ring in which for every element $x \in R$ there exists *a* (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. For all elements x, y, z of R

$$y \leq z \Rightarrow xy \leq xz \tag{iv}$$

and
$$x^{n(x)-1}.\, y \leq y \tag{v}$$

Proof: Let x, y, z be any three elements of R. In view of (iii) $y \leq z \Rightarrow yz = y^2 \Rightarrow x^2(yz) = x^2y^2 \Rightarrow (xy)(xz) = (xy)^2 \Rightarrow xy \leq xz$.

Further, in view of (i) we have $x^{n(x)-1}.\, y^2 = (x^{n(x)-1}\, y)^2 \Rightarrow x^{n(x)-1}.\, y \leq y$.

Definition 2.3: Let R be a ring in which for every element $x \in$ R there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. A nonzero element a of R is called an atom of R provided for every $x \in$ R,

$$x \leq a \text{ implies } x = a \text{ or } x = 0 \qquad \text{(vi)}$$

More over, R is called atomic provided for every nonzero element r of R there exists an atom a of R such that $a \leq r$.

Lemma 2.4: Let R be a ring in which for every element $x \in$ R there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$, and let a be an atom of R. Then $r^{n(r)-1} . a = a$ or $ra=0$, for every element r of R.

Proof: By (v),we have $r^{n(r)-1}a \leq a$ and since a is an atom by (vi) we have $r^{n(r)-1}a = a$ or $r^{n(r)-1}a = 0$. $\Rightarrow$ $r^{n(r)-1}a = a$ or $ra=0$ (since $r^{n(r)} = r$).

Definition 2.5: Let R be a ring in which for every element $x \in$ R there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. A subset S of R is called orthogonal provided $xy = 0$ for distinct elements x and y of S.

Lemma 2.6: Let R be a ring in which for every element $x \in$ R there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. Then the set $(e_i)_{i \in I}$ of all idempotent atoms of R is an orthogonal set.

Proof: Since for each i $\in$ I, e_i is both an atom and an idempotent, from Lemma (2.4) it follows that $e_i e_j = e_j = e_i$ or $e_i e_j = 0$.

Lemma 2.7: Let R be a ring in which for every element $x \in$ R there exists *a* (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$, and let a be an atom of R. Then $a^{n(a)-1}$ is an idempotent atom of R.

Proof: From (i) it follows that $a^{n(a)-1}$ is idempotent. Now, let $x \leq a^{n(a)-1}$. But by (iv) we get $ax \leq a^{n(a)} = a$ i.e., $ax \leq a$. Since a is an atom by (vi) it follows that $ax = a$ or $ax = 0$.

If $ax = a$ then $a^{n(a)-1} \cdot x = a^{n(a)-1}$. By (iii) we get $a^{n(a)-1} \leq x$. Hence, $x = a^{n(a)-1}$.

If $ax = 0$ then $a^{n(a)-1} \cdot x = 0$; but $a^{n(a)-1} \cdot x = x^2$. Therefore $x^2 = 0$. By (ii) we get $x = 0$.

Lemma 2.8: Let R be a ring in which for every element $x \in$ R there exists *a* (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$, and let $(e_i)_{i \in I}$ be the set of all idempotent atoms of R, then for every i $\in$ I the ideal F_i of R given by

$$F_i = \{r\, e_i \,/\, r \in R\} \qquad \text{(vii)}$$

is a subfield of R.

Proof: Since $e_i^2 = e_i$, it follows that e_i is an element of F_i and also the unit of F_i.

Now let re_i be a non zero element of F_i. We show that re_i has an inverse in F_i. If $n(r) > 2$ then by lemma (2.4) we have $(r\ e_i)\ (r^{n(r)-2}e_i) = e_i$. It follows that $r^{n(r)-2}e_i$ is the inverse of re_i in F_i.

If $n(r) = 2$ then by lemma (2.4) we have $(r\ e_i)\ (r\ e_i) = r^2\ e_i^2 = re_i = e_i$. It shows that re_i has its own inverse in F_i.

Now, we are ready to prove the main theorem.

Theorem 2.9: The ring R in which for every element $x \in R$ there exists *a* (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is always a Smarandache ring.

Proof: Let $(e_i)_{i \in I}$ be the set of all idempotent atoms of R. In view of the lemma (2.8), for every $i \in I$, the ideal F_i of R given by $F_i = \{re_i \ / \ r \in R\}$ is a field of R. Hence, the ring R is a Smarandache ring.

SECTION – 3

In this section we give examples to justify our theorem (2.9). Further, we show by an example that the condition 'for every element $x \in R$ there exists *a* (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$' satisfied by the ring R in our results is a sufficient condition but not a necessary condition.

Example 3.1: Consider the ring Z_{10} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}(modulo integers). It is obvious that $0^2 = 0$; $1^2 = 1$; $2^5 = 2$; $3^5 = 3$; $4^3 = 4$; $5^2 = 5$; $6^2 = 6$; $7^5 = 7$; $8^5 = 8$; $9^3 = 9$. Therefore the ring R = Z_{10} satisfies the condition 'for every element $x \in$ R there exists *a* (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$'. Further, in view of the relation table (see table I), and lemma (2.1), (Z_{10}, $\leq$) is a partially ordered set. The Hasse diagram (see [9]) of the p.o. set (Z_{10}, $\leq$) is given (see fig. I) for our use.

From the Hasse diagram it is obvious that the elements 2, 4, 5, 6, 8 are atoms and the elements 5, 6 are idempotent atoms in (Z_{10}, $\leq$). In view of lemma (2.8), the ideals

$F_1 = \{r.5 \,/\, r \in Z_{10}\} = \{0, 5\}$ and

$F_2 = \{r.6 \,/\, r \in Z_{10}\} = \{0,2,4,6,8\}$ are fields. Hence the ring Z_{10} (modulo integers) is a Smarandache ring.

≤	0	1	2	3	4	5	6	7	8	9
0	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
1		✓								
2			✓					✓		
3				✓						
4					✓					✓
5		✓		✓		✓		✓		✓
6		✓					✓			
7								✓		
8				✓					✓	
9										✓

Table – I

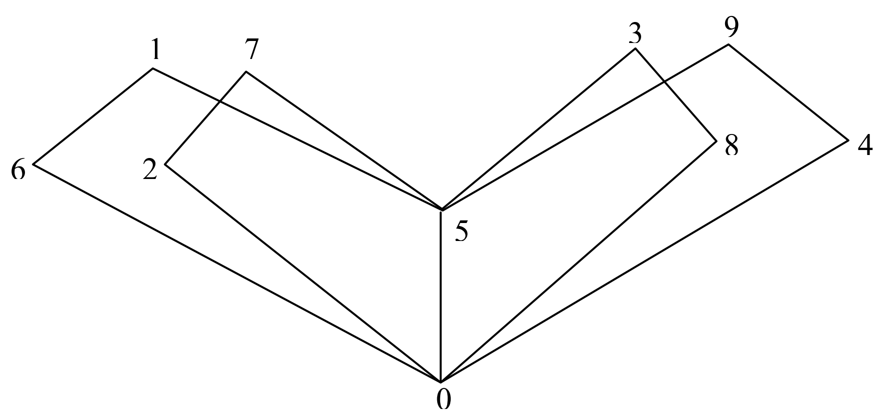

Fig – I

Example 3.2: Consider the ring Z_6 = {0, 1, 2, 3, 4, 5} (modulo integers).It is obvious that $0^2 = 0$; $1^2 = 1$; $2^3 = 2$; $3^2 = 3$; $4^2 = 4$; $5^3 = 5$. Therefore, the ring R= Z_6 satisfies the condition 'for every element $x \in$ R there exists *a* (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$'. In view of the relation table (see table II), and lemma (2.1), (Z_6, ≤) is a partially ordered set. The Hasse diagram (see [9]) of the p.o. set (Z_6, ≤) is given (see fig. II) for our use.

From the Hasse diagram, it is obvious that the elements 2, 3,4, are atoms and the elements 3, 4 are idempotent atoms in (Z_6, ≤). In view of lemma (2.8), the ideals $F_1 = \{r.3 \,/\, r \in Z_6\} = \{0, 3\}$ and $F_2 = \{r.4 \,/ r \in Z_6\} = \{0, 2, 4,\}$ are fields. Hence Z_6 (modulo integers) is a Smarandache ring.

≤	0	1	2	3	4	5
0	✓	✓	✓	✓	✓	✓
1		✓				
2			✓			✓
3		✓		✓		✓
4		✓			✓	
5						✓

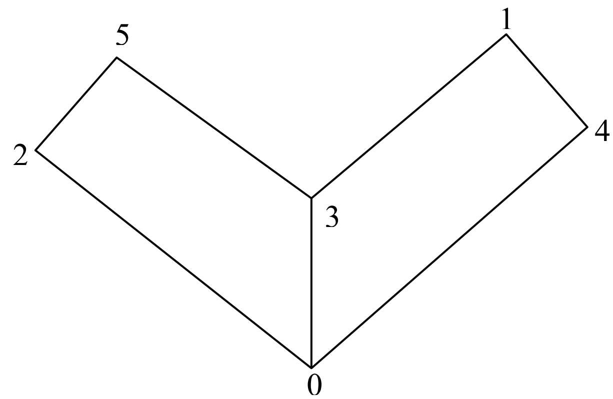

Fig – II

Finally, we show by an example that the condition 'for every element $x \in$ R there exists *a* (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$' satisfied by the ring R in our results is a sufficient condition but not a necessary condition.

Example 3.3: In [13] Vasantha Kandasamy W.B. quoted the example (1.3) for Smarandache ring. This ring Z_{12} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} (modulo integers) is a Smarandache ring but the condition ' for every element $x \in$ R there exists *a* (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$' fails in the ring Z_{12} as there does not exist an integer n(2) > 1 for the integer 2 in Z_{12} such that $2^{n(2)} = 2$. Hence, the condition is a sufficient condition but not a necessary condition.

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