Why a Minimal Length follows from the Extended Relativity Principle in Clifford Spaces

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Abstract

Recently, novel physical consequences of the Extended Relativity Theory in C-spaces (Clifford spaces) were explored and which provided a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework. An elegant *nonlinear* momentum-addition law was derived that tackled the "soccer-ball" problem in DSR. Generalized photon dispersion relations allowed also for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary spacetimes, but breaking the *extended* Lorentz symmetry in C-spaces. This does *not* occur in DSR nor in other approaches, like the presence of quantum spacetime foam. In this work we show why a *minimal* length (say the Planck scale) follows naturally from the Extended Relativity principle in Clifford Spaces. Our argument relies entirely on the Physics behind the extended notion of Lorentz transformations in C-space, and *does not* invoke quantum gravity arguments, nor quantum group deformations of Lorentz/Poincare algebras, nor other prior arguments displayed in the Physics literature. The Extended Relativity Theory in Clifford *Phase* Spaces requires also the introduction of a *maximal* scale which can be identified with the Hubble scale. It is found also that C-space physics favors a choice of signature $(-, +, +, ..., +)$.

Keywords : Clifford algebras; Extended Relativity in Clifford Spaces; Doubly Special Relativity; Quantum Clifford-Hopf algebras.

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[*] Dedicated to the memory of Rachael Bowers

1 Introduction

1.1 Novel Consequences of the Extended Relativity Theory in Clifford Spaces

In the past years, the Extended Relativity Theory in C-spaces (Clifford spaces) and Clifford-Phase spaces were developed [1], [2]. The Extended Relativity theory in Clifford-spaces (C-spaces) is a natural extension of the ordinary Relativity theory whose generalized coordinates are Clifford polyvector-valued quantities which incorporate the lines, areas, volumes, and hyper-volumes degrees of freedom associated with the collective dynamics of particles, strings, membranes, p-branes (closed p-branes) moving in a D-dimensional target spacetime background. C-space Relativity permits to study the dynamics of all (closed) p-branes, for different values of p, on a unified footing. Our theory has 2 fundamental parameters : the speed of a light c and a length scale which can be set equal to the Planck length. The role of "photons" in C-space is played by *tensionless* branes. An extensive review of the Extended Relativity Theory in Clifford spaces can be found in [1]. The polyvector valued coordinates $x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, ...$ are now linked to the basis vectors generators γ^μ, bi-vectors generators $\gamma_\mu \wedge \gamma_\nu$, tri-vectors generators

$\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$, ... of the Clifford algebra, including the Clifford algebra unit element (associated to a scalar coordinate). These polyvector valued coordinates can be interpreted as the quenched-degrees of freedom of an ensemble of p-loops associated with the dynamics of closed p-branes, for $p = 0, 1, 2, ..., D-1$, embedded in a target D-dimensional spacetime background.

The C-space polyvector-valued momentum is defined as $\mathbf{P} = d\mathbf{X}/d\Sigma$ where $\mathbf{X}$ is the Clifford-valued coordinate corresponding to the $Cl(1,3)$ algebra in four-dimensions, for example,

$$\mathbf{X} \ = \ s\ \mathbf{1} \ + \ x^\mu\ \gamma_\mu \ + \ x^{\mu\nu}\ \gamma_\mu \wedge \gamma_\nu \ + \ x^{\mu\nu\rho}\ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \ + \ x^{\mu\nu\rho\tau}\ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\tau \quad (1)$$

where we have omitted combinatorial numerical factors for convenience in the expansion (1). It can be generalized to any dimensions, including $D = 0$. The component s is the Clifford scalar component of the polyvector-valued coordinate and $d\Sigma$ is the infinitesimal C-space proper "time" interval which is *invariant* under $Cl(1,3)$ transformations which are the Clifford-algebra extensions of the $SO(1,3)$ Lorentz transformations [1]. One should emphasize that $d\Sigma$, which is given by the square root of the quadratic interval in C-space

$$(d\Sigma)^2 \ = \ (ds)^2 \ + \ dx_\mu\ dx^\mu \ + \ dx_{\mu\nu}\ dx^{\mu\nu} \ + \ \ldots \quad (2)$$

is *not* equal to the proper time Lorentz-invariant interval $d\tau$ in ordinary spacetime $(d\tau)^2 = g_{\mu\nu}dx^\mu dx^\nu = dx_\mu dx^\mu$. In order to match units in all terms of eqs-(1,2) suitable powers of a length scale (say Planck scale) must be introduced. For convenience purposes it is can be set to unity. For extensive details of the generalized Lorentz transformations (poly-rotations) in flat C-spaces and references we refer to [1].

Let us now consider a basis in C-space given by

$$E_A = \gamma,\ \gamma_\mu,\ \gamma_\mu \wedge \gamma_\nu,\ \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho,\ ... \qquad (3)$$

where γ is the unit element of the Clifford algebra that we label as $\mathbf{1}$ from now on. In (3) when one writes an r-vector basis $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices in "lexicographical" order so that $\mu_1 < \mu_2 < .... < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X \ = \ X^A E_A \ = \ s\,\mathbf{1} \ + \ x^\mu \gamma_\mu \ + \ x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \qquad (4)$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, ..., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops,..., onto the coordinate planes . By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string. In order to avoid using the powers of the Planck scale length parameter L_p in the expansion of the polyvector X (in order to match units) we can set it to unity to simplify matters. In a *flat* C-space the basis vectors E^A, E_A are *constants*. In a *curved* C-space this is no longer true. Each E^A, E_A is a function of the C-space coordinates

$$X^A \ = \ \{ \ s,\ x^\mu,\ x^{\mu_1\mu_2},\ .....,\ x^{\mu_1\mu_2.....\mu_D} \ \} \qquad (5)$$

which include scalar, vector, bivector,..., p-vector,... coordinates in the underlying D-dim base spacetime and whose corresponding C-space is 2^D-dimensional since the Clifford algebra in D-dim is 2^D-dimensional.

Defining

$$E^A \equiv \gamma^A,\ \ \mathcal{J}^{AB} \equiv \ \frac{1}{2}\,(\gamma^A \otimes \gamma^B - \gamma^B \otimes \gamma^A),\ \ \mathcal{J}^A \ \equiv \ \frac{1}{2}\,(\gamma^A \otimes \mathbf{1} - \mathbf{1} \otimes \gamma^A) \ \neq \ 0 \qquad (6)$$

for arbitrary polyvector valued indices $A, B, ...$ and after using the relations

$$[\ \gamma^A \otimes \gamma^B,\ \gamma^C \otimes \gamma^D\ ] \ = \ \frac{1}{2}\,[\ \gamma^A,\ \gamma^C\ ] \otimes \{\gamma^B,\ \gamma^D\ \} \ + \ \frac{1}{2}\,\{\ \gamma^A,\ \gamma^C\ \} \otimes [\ \gamma^B,\ \gamma^D\ ] \qquad (7)$$

$$\{\ \gamma^A \otimes \gamma^B,\ \gamma^C \otimes \gamma^D\ \} \ = \ \frac{1}{2}\,[\ \gamma^A,\ \gamma^C\ ] \otimes [\ \gamma^B,\ \gamma^D\ ] \ + \ \frac{1}{2}\,\{\ \gamma^A,\ \gamma^C\ \} \otimes \{\ \gamma^B,\ \gamma^D\ \} \qquad (8)$$

yields, for example, the commutator relation involving the boost generator $\mathcal{J}^{01}$ (along the X_1 direction) and the area-boost generator $\mathcal{J}^{0\ 12}$ (along the bivector X_{12} direction) in C-space

$$[\ \mathcal{J}^{0\ 12},\ \mathcal{J}^{01}\ ] \ = \ \frac{1}{4}\,[\ \gamma^0 \otimes \gamma^{12} - \gamma^{12} \otimes \gamma^0,\ \gamma^{01} \otimes \mathbf{1} - \mathbf{1} \otimes \gamma^{01}\ ] \ =$$

$$-\ \frac{1}{8}\,g^{11}\,(\gamma^{20} \otimes \gamma^0 - \gamma^0 \otimes \gamma^{20}) \ - \ \frac{1}{8}\,g^{00}(\gamma^1 \otimes \gamma^{12} - \gamma^{12} \otimes \gamma^1) \qquad (9)$$

The (anti) commutators of all the gamma generators are explicitly given in the Appendix. One requires to use the expressions in the Appendix in order to arrive at the last terms of eq-(9). Hence, from the definitions in eqs-(6) one learns that

$$[\ \mathcal{J}^{0\ 12},\ \mathcal{J}^{01}\ ]\ =\ \frac{1}{4}\ g^{00}\ \mathcal{J}^{12\ 1}\ +\frac{1}{4}\ g^{11}\ \mathcal{J}^{02\ 0} \tag{10}$$

A careful study reveals that the commutators obtained in eq-(10) (after using the expressions in eqs-(7,8) and in those in the Appendix) *do not obey* the relations

$$[\ \mathcal{J}^{AB},\ \mathcal{J}^{C}\ ]\ \sim\ -\ G^{AC}\ \mathcal{J}^{B}\ +\ G^{BC}\ \mathcal{J}^{A} \tag{11}$$

$$[\ \mathcal{J}^{AB},\ \mathcal{J}^{CD}\ ]\ \sim\ -\ G^{AC}\ \mathcal{J}^{BD}\ +\ G^{AD}\ \mathcal{J}^{BC}\ -\ G^{BD}\ \mathcal{J}^{AC}\ +\ G^{BC}\ \mathcal{J}^{AD} \tag{12}$$

where the C-space metric is chosen to be $G^{AB} = 0$ when the *grade* $A \neq$ *grade* B. And for the same-grade metric components $g^{[a_1 a_2 ... a_k]\ [b_1 b_2 ... b_k]}$ of G^{AB}, the metric can decomposed into its irreducible factors as antisymmetrized sums of products of η^{ab} given by the following *determinant* [17]

$$G^{AB}\ \equiv\ det\ \begin{pmatrix} \eta^{a_1 b_1} & \ldots & \ldots & \eta^{a_1 b_k} \\ \eta^{a_2 b_1} & \ldots & \ldots & \eta^{a_2 b_k} \\ ------------ & & --------------- & \\ \eta^{a_k b_1} & \ldots & \ldots & \eta^{a_k b_k} \end{pmatrix} \tag{13}$$

The spacetime signature is chosen to be $(-, +, +, ..., +)$.

It would be tempting to suggest that the C-space generalization of the Poincare algebra could be given by the commutators in eq-(12) and

$$[\ \mathcal{J}^{AB},\ P^{C}\ ]\ \sim\ -\ G^{AC}\ P^{B}\ +\ G^{BC}\ P^{A},\ \ [P^A,\ P^B]\ =\ 0 \tag{14}$$

where P^A is the poly-momentum and $\mathcal{J}^{AB}$ are the generalized Lorentz generators. A more careful inspection suggests that this *is not* the case. The actual commutators are more complicated as displayed by eq-(10). One always must use the relations in eqs-(7,8) and in the Appendix in order to determine the $[\mathcal{J}^{AB}, \mathcal{J}^{CD}], [\mathcal{J}^{AB}, \mathcal{J}^{C}]$ commutators. In this way one will ensure that the Jacobi identities are satisfied.

Let us provide some examples of the generalized Lorentz relativistic transformations in C-space. Performing an area-boost transformation along the bivector X_{12} direction and followed by a boost transformation along the X_1 direction one arrives after some laborious but straightforward algebra at

$$X''_0\ =\ (\ X_0\ cosh\beta\ + L^{-1}\ X_{12}\ sinh\beta\ )\ cosh\alpha\ +\ X_1\ sinh\alpha \tag{15a}$$

$$X''_1\ =\ (\ X_0\ cosh\beta\ + L^{-1}\ X_{12}\ sinh\beta\ )\ sinh\alpha\ +\ X_1\ cosh\alpha \tag{15b}$$

$$X''_{12}\ =\ L\ X_0\ sinh\beta\ +\ X_{12}\ cosh\beta \tag{15c}$$

$$X''_2\ =\ X_2,\ \ X''_{01}\ =\ X_{01},\ \ X''_{02}\ =\ X_{02},\ \ X''_{012}\ =\ X_{012} \tag{15d}$$

the Clifford scalar parts of the polyvectors are trivially invariant $s'' = s$ as they should. The parameter α is the standard Lorentz boost parameter and β is the area-boosts one.

Due to the identities $cosh^2\alpha - sinh^2\alpha = 1$ and $cosh^2\beta - sinh^2\beta = 1$, a straightforward algebra leads to

$$- (X''_0)^2 + (X''_1)^2 + L^{-2} (X''_{12})^2 = - (X_0)^2 + (X_1)^2 + L^{-2} (X_{12})^2 \qquad (16)$$

which is a consequence of the invariance of the norm in C-space [1]

$$< \mathbf{X}^\dagger \mathbf{X} > = X_A X^A = s^2 + X_\mu X^\mu + X_{\mu_1\mu_2} X^{\mu_1\mu_2} + ... X_{\mu_1\mu_2.....\mu_D} X^{\mu_1\mu_2....\mu_D} \quad (17)$$

where $\mathbf{X}^\dagger$ denotes the reversal operation obtained by reversing the order of the gamma generators in the wedge products. The symbol $< .... >$ denotes taking the scalar part in the Clifford geometric product.

In the particular case when the spacetime dimension is chosen to be $D = 3$ for simplicity, one has in addition to the transformations provided by eqs-(15) that the other remaining coordinates remain invariant under boosts along the X_1 direction and area-boosts along X_{12},

Performing, instead, a boost transformation along the X_1 direction and then followed by an area-boost transformation along the bivector X_{12} direction one arrives at

$$X''_0 = (X_0\, cosh\alpha + X_1\, sinh\alpha)\, cosh\beta + L^{-1} X_{12}\, sinh\beta \qquad (18a)$$

$$X''_1 = X_0\, sinh\alpha + X_1\, cosh\alpha \qquad (18b)$$

$$X''_{12} = X_{12}\, cosh\beta + L\, (X_0\, cosh\alpha + X_1\, sinh\alpha)\, sinh\beta \qquad (18c)$$

straightforward algebra leads again to

$$- (X''_0)^2 + (X''_1)^2 + L^{-2} (X''_{12})^2 = - (X_0)^2 + (X_1)^2 + L^{-2} (X_{12})^2 \qquad (19)$$

We may notice the *mixing* of polyvector valued coordinates under generalized Lorentz transformations in C-space. Stringy (area coordinates) $X^{\mu\nu}$ and point particle coordinates X^μ in eqs-(15,18) appear mixed with each other under the C-space transformations.

Because $[\ \mathcal{J}^{0\ 12},\ J^{01}\ ]\ \neq 0$ the *order* in which one performs the generalized boosts transformations matters. In ordinary Relativity the commutator of two boosts $[M^{0i}, M^{0j}] \sim \eta^{00}\ M^{ij}$ gives a rotation. This is the reasoning behind the Thomas precession. In C-space, one will arrive at different results if one first performs an area-boost followed by an ordinary boost, compared if we perform an ordinary boost followed by an area boost. This is the reason why eqs-(15) $differ$ from eqs-(18) although both of them lead to the same invariance property of the C-space interval (17) .

There are more general areal X_{12} boosts transformations defined in terms of two parameters α, β and involving the temporal bivector X_{01} and temporal trivector X_{012} coordinates as follows

$$X'_{12} = (X_{12}\, cosh\alpha + L^{-1} X_{012}\, sinh\alpha)\, cosh\beta + X_{01}\, sinh\beta \qquad (20a)$$

$$X'_{01} = X_{01}\, cosh\beta + (X_{12}\, cosh\alpha + L^{-1} X_{012}\, sinh\alpha)\, sinh\beta \qquad (20b)$$

$$X'_{012} = X_{012}\, cosh\alpha + L\, X_{12}\, sinh\alpha$$

$$X'_0 \ = \ X_0, \ \ X'_1 \ = \ X_1, \ \ X'_2 \ = \ X_2, \ \ X'_{02} \ = \ X_{02} \qquad (20c)$$

the transformations leave *invariant* the following subinterval of the full interval in C-space when the $3D$ spacetime signature is $(-,+,+)$

$$L^2 \ (X'_{12})^2 \ - L^2 \ (X'_{01})^2 \ - \ (X'_{012})^2 = \ L^2 \ (X_{12})^2 \ - L^2 \ (X_{01})^2 \ - \ (X_{012})^2 \qquad (20c)$$

we may notice that the spatial variables and temporal ones appear with opposite signs in (20d) , as they should, and hence the transformations in eqs-(20) are valid boosts.

The C-space *rotations* like those mixing the area-bivector X^{12} with the X^1 vector component are of the form

$$X'_1 \ = \ X_1 \ cos\theta \ - L^{-1} \ X_{12} \ sin\theta; \ \ X'_{12} \ = \ L \ X_1 \ sin\theta \ + \ X_{12} \ cos\theta \qquad (21)$$

such that

$$L^{-2} \ (X'_{12})^2 \ + \ (X'_1)^2 \ = \ L^{-2} \ (X_{12})^2 \ + \ (X_1)^2 \qquad (22)$$

Due to the fact that $g^{11} = g^{22} = 1$ this explains why $(X_{12})^2$ appears with a plus sign in all the above equations.

Recently, novel physical consequences of the Extended Relativity Theory in C-spaces (Clifford spaces) were explored in [4]. The latter theory provides a very different physical explanation of the phenomenon of "relativity of locality" than the one described by the Doubly Special Relativity (DSR) framework. Furthermore, an elegant *nonlinear* momentum-addition law was derived in order to tackle the "soccer-ball" problem in DSR. Neither derivation in C-spaces requires a *curved* momentum space nor a deformation of the Lorentz algebra. While the constant (energy-independent) speed of photon propagation is always compatible with the generalized photon dispersion relations in C-spaces, another important consequence was that the generalized C-space photon dispersion relations allowed also for energy-dependent speeds of propagation while still *retaining* the Lorentz symmetry in ordinary spacetimes, while breaking the *extended* Lorentz symmetry in C-spaces. This does *not* occur in DSR nor in other approaches, like the presence of quantum spacetime foam.

We learnt from Special Relativity that the concept of simultaneity is also relative. By the same token, we have shown in [4] that the concept of spacetime locality is *relative* due to the *mixing* of area-bivector coordinates with spacetime vector coordinates under generalized Lorentz transformations in C-space. In the most general case, there will be mixing of all polyvector valued coordinates. This was the motivation to build a unified theory of all extended objects, p-branes, for all values of p subject to the condition $p+1 = D$.

In [5] we explored the many novel physical consequences of Born's Reciprocal Relativity theory [7], [9], [10] in flat phase-space and generalized the theory to the curved phase-space scenario. We provided six specific novel physical results resulting from Born's Reciprocal Relativity and which are *not* present in Special Relativity. These were : momentum-dependent time delay in the emission and detection of photons; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations. We finalized by constructing a Born reciprocal general relativity theory in curved phase-spaces which required the introduction of a complex Hermitian metric, torsion and nonmetricity.

We should emphasize that *no* spacetime foam was introduced, nor Lorentz invariance was broken, in order to explain the time delay in the photon emission/arrival. In the conventional approaches of DSR (Double Special Relativity) where there is a Lorentz invariance breakdown [13], a longer wavelength photon (lower energy) experiences a smoother spacetime than a shorter wavelength photon (higher energy) because the higher energy photon experiences more of the graininess/foamy structure of spacetime at shorter scales. Consequently, the less energetic photons will move faster (less impeded) than the higher energetic ones and will arrive at earlier times.

However, in our case above [5] the time delay is entirely due to the very nature of Born's Reciprocal Relativity when one looks at pure acceleration (force) boosts transformations of the phase space coordinates in $flat$ phase-space. No *curved* momentum space is required as it happens in [13]. The time delay condition in Born's Reciprocal Relativity theory implied also that higher momentum (higher energy) photons will take longer to arrive than the lower momentum (lower energy) ones.

Superluminal particles were studied within the framework of the Extended Relativity theory in Clifford spaces (C-spaces) in [6]. In the simplest scenario, it was found that it is the contribution of the Clifford scalar component P of the poly-vector-valued momentum $\mathbf{P}$ which is responsible for the superluminal behavior in ordinary spacetime due to the fact that the effective mass $\sqrt{\mathcal{M}^2 - P^2}$ can be imaginary (tachyonic). However from the point of view of C-space there is no superluminal behaviour (tachyonic) because the true physical mass still obeys $\mathcal{M}^2 > 0$. As discussed in detailed by [1], [3] one can have tachyonic (superluminal) behavior in ordinary spacetime while having non-tachyonic behavior in C-space. Hence from the C-space point of view there is no violation of causality nor the Clifford-extended Lorentz symmetry. The analog of "photons" in C-space are *tensionless* strings and branes [1].

Long ago [18] we showed how the quadratic Casimir invariant in C-space given by eq-(25) leads to modified wave equations, dispersion laws and to the generalizations of the stringy-uncertainty principle relations. One is able to arrive at the energy-dependent speed of propagation while still *retaining* the Lorentz symmetry. This does *not* occur in DSR nor in other approaches. For further details we refer to [4].

2 On Areal Velocities, Multiple Times and Minimal Length

2.1 Addition Law of Areal Velocities

Setting $\alpha = 0$ in eqs-(15,18) and differentiating gives

$$dX''_0 \ = \ dX_0 \ cosh\beta \ + L^{-1} \ dX_{12} \ sinh\beta, \ \ dX''_1 \ = \ dX_1 \tag{23a}$$

$$dX''_{12} \ = \ dX_{12} \ cosh\beta \ + L \ dX_0 \ sinh\beta \tag{23b}$$

such that

$$\frac{dX''_{12}}{dX''_0} \ = \ \frac{dX_{12} \ cosh\beta \ + L \ dX_0 \ sinh\beta}{dX_0 \ cosh\beta \ + L^{-1} \ dX_{12} \ sinh\beta} \ = \ \frac{\frac{dX_{12}}{dX_0} \ + \ L \ tanh\beta}{1 + L^{-1} \ \frac{dX_{12}}{dX_0} \ tanh\beta} \tag{23c}$$

Using the following definitions of the areal velocities (in $c = 1$ units)

$$V_{12} \ \equiv \ \frac{dX_{12}}{dX_0}, \ \ V'_{12} \ \equiv \ L \ tanh\beta \tag{24}$$

corresponding to the areal velocity of a polyparticle as measured in a given frame of reference, and the areal velocity associated with an areal boost transformation of the reference frame, respectively, one can rewrite eq-(23c) as

$$V''_{12} \ = \ \frac{V_{12} \ + \ V'_{12}}{1 \ + \ \frac{V_{12} \ V'_{12}}{L^2}} \tag{25}$$

leading to the *addition* law of the *areal* velocities. In particular, one can see that if the *maximal* areal velocity is identified with the quantity Lc, after restoring the speed of light that was set to unity, we have that the addition/subtraction law of the maximal areal velocities Lc yields always the maximal areal velocity

$$V''_{12} \ = \ \frac{V_{12} \ \pm \ V'_{12}}{1 \ \pm \ \frac{V_{12} \ V'_{12}}{L^2c^2}} \ = \ \frac{Lc \ \pm \ Lc}{1 \ \pm \ \frac{Lc \ Lc}{L^2c^2}} \ = \ Lc \ \frac{1 \ \pm \ 1}{1 \ \pm \ 1} \ = \ Lc \tag{26}$$

so that the *maximal* areal velocity Lc is never surpassed and it is a C-space relativistic *invariant* quantity. Meaning that if the areal velocities of two polyparticles in a given reference frame is maximal Lc, their relative areal-velocity is also maximal Lc and is obtained from the subtraction law in eq-(26).

Let us take the spacetime signature to be $(-, +, +, +, ..., +)$ and factorize the C-space interval (2) as follows by bringing the $c^2(dt)^2$ factor outside the parenthesis

$$(d\Sigma)^2 \ = \ c^2(dt)^2 \ \left(\ \frac{L^2}{c^2} \ (\frac{ds}{dt})^2 \ - \ 1 \ + \ \frac{1}{c^2} \ (\frac{dX_i}{dt})^2 \ + \frac{1}{L^2c^2} \ (\frac{dX_{ij}}{dt})^2 \ - \ \frac{1}{L^2c^2} \ (\frac{dX_{0i}}{dt})^2 \ .... \right) \tag{27}$$

where the spatial index i range is $1, 2, ..., D-1$. The Clifford space associated with the Clifford algebra in $4D$ is 16-dimensional and has a neutral/split signature of $(8, 8)$ [3], [1]. For example, the terms $(dX_{0i})^2, (dX_{0ij})^2, (dX_{0123})^2$ will appear with a negative sign, while the terms $(dX_{ij})^2, (dX_{ijk})^2$ will appear with a positive sign.

There are many possible combination of numerical values for the 16 terms inside the parenthesis in eq-(27). As explained in [3], [1], *superluminal* velocities in ordinary spacetime are possible, while retaining the null interval condition in C-space $(d\Sigma)^2 = 0$, associated with *tensionless* branes. The null interval in C-space $(d\Sigma)^2 = 0$ can be attained, for example, if each term inside the parenthesis is ± 1 respectively. Since there are 8 positive ($+1$) terms and 8 negative (-1) terms one has that $8 - 8 = 0$ and the null interval condition $(d\Sigma)^2 = 0$ is still satisfied despite having superluminal speeds.

A very different combination of numerical values, as compared to the previous one, leading also to a null interval condition in C-space $(d\Sigma)^2 = 0$, occurs when one does *not exceed* the maximal *magnitudes* for the linear, areal, volume, ... velocities. It is given by the following combination

$$\frac{1}{c^2} \left((\frac{dX_1}{dt})^2 + (\frac{dX_2}{dt})^2 + (\frac{dX_3}{dt})^2 \right) = 1 \tag{28a}$$

$$\frac{1}{L^2 c^2} \left((\frac{dX_{12}}{dt})^2 + (\frac{dX_{13}}{dt})^2 + (\frac{dX_{23}}{dt})^2 \right) =$$

$$\frac{1}{L^2 c^2} \left((\frac{dX_{01}}{dt})^2 + (\frac{dX_{02}}{dt})^2 + (\frac{dX_{03}}{dt})^2 \right) = 1 \tag{28b}$$

$$\frac{1}{L^4 c^2} \left((\frac{dX_{012}}{dt})^2 + (\frac{dX_{013}}{dt})^2 + (\frac{dX_{023}}{dt})^2 \right) = \frac{1}{L^4 c^2} (\frac{dX_{123}}{dt})^2 = 1 \tag{28c}$$

$$\frac{1}{L^6 c^2} (\frac{dX_{0123}}{dt})^2 = 1, \quad \frac{L^2}{c^2} (\frac{ds}{dt})^2 = 1 \tag{28d}$$

In this fashion from eqs-(28) one can have the analog of a C-space "photon" which corresponds to a polyparticle whose *magnitudes* of the spatial and temporal components of the linear, areal, volume, ... velocities are respectively given by their *maximal* values $c, Lc, L^2c, L^3c, ...$ One must also include the velocity component associated with the Clifford scalar component of the polyvector given by ds/dt and whose maximal value is set to be c/L.

- **Effective Time and Another Description of C-space Photons**

From now on, in order to simplify matters let us work with $D = 3$ instead of $D = 4$. The effective temporal variable T is defined as

$$c^2(dT)^2 \equiv c^2(dt)^2 + \frac{1}{c^2} (\frac{dX_{01}}{dt})^2 + \frac{1}{c^2} (\frac{dX_{02}}{dt})^2 + \frac{1}{L^2 c^2} (\frac{dX_{012}}{dt})^2 \tag{29}$$

so that the C-space interval can be rewritten, after factoring out the $c^2(dT)^2$ term, as

$$(d\Sigma)^2 \ = \ - \ c^2(dT)^2 \ \left(1 \ - \ \frac{L^2}{c^2} \ (\frac{ds}{dT})^2 \ - \ \frac{1}{c^2} \ (\frac{dX_1}{dT})^2 \ - \ \frac{1}{c^2} \ (\frac{dX_2}{dT})^2 \ - \ \frac{1}{L^2c^2} \ (\frac{dX_{12}}{dT})^2 \right) \tag{30}$$

The last expression has the same functional form as the ordinary spacetime interval in MInkowski space. Namely one can write the C-space interval $(d\Sigma)^2$ in the form

$$(d\Sigma)^2 \ = \ - \ c^2(dT)^2 \ (\ 1 \ - \ \frac{V^2}{c^2}) \tag{31}$$

where the generalization of the magnitude-squared of the spatial velocity divided by c^2 is

$$\frac{V^2}{c^2} \ \equiv \ \frac{L^2}{c^2} \ (\frac{ds}{dT})^2 \ + \ \frac{1}{c^2} \ (\frac{dX_1}{dT})^2 \ + \ \frac{1}{c^2} \ (\frac{dX_2}{dT})^2 \ + \ \frac{1}{L^2c^2} \ (\frac{dX_{12}}{dT})^2 \tag{32}$$

Another description of C-space Photons is obtained from the null C-space interval condition $(d\Sigma)^2 = 0$ which is equivalent to setting $V^2/c^2 = 1$ in eq-(32) and where the velocity squared is defined with respect to the effective temporal variable T.

2.2 On Minimal Length from Addition Law of Areal Velocities

An ordinary spacetime boost along the spatial direction X^1 can be seen as a "rotation" in the $X^0 - X^1$ plane. Rotations with a purely imaginary angle behave like boosts. Therefore, a more general areal-boost transformation should be such that it "rotates" the spatial areal-bivector coordinate X_{12} into a linear combination of the ordinary temporal coordinate X_0, the temporal bivector coordinates X_{01}, X_{02}, *and* the temporal trivector coordinate X_{012}. A simplified areal boost transformation is given, for example, by

$$\begin{aligned} X'_{12} \ &= \ X_{12} \ cosh\beta \ + \ X_{01} \ sinh\beta \\ X'_{01} \ &= \ X_{01} \ cosh\beta \ + \ X_{12} \ sinh\beta \\ X'_0 \ = \ X_0, \ \ X'_1 \ &= \ X_1, \ \ X'_2 \ = \ X_2 \\ X'_{02} \ = \ X_{02}, \ \ X_{012} \ &= \ X_{012} \end{aligned} \tag{33}$$

In doing so the subinterval

$$(X'_{12})^2 \ - \ (X'_{01})^2 \ = \ (X_{12})^2 \ - \ (X_{01})^2 \tag{34}$$

remains invariant, irrespective of the spacetime signature. In this special case for eq-(34), changing the spacetime signature does *not* alter the overall signs in eq-(34). The signs remain the same whether the signature is $(-, +, +)$ or $(+, -, -)$. This is not the case in general as we shall see.

The areal-velocity is now defined with respect to the temporal bivector coordinate X_{01}, instead of the ordinary temporal variable X_0,

$$\frac{dX'_{12}}{dX'_{01}} = \frac{dX_{12}\ cosh\beta\ +\ dX_{01}\ sinh\beta}{dX_{01}\ cosh\beta\ +\ dX_{12}\ sinh\beta} = \frac{\frac{dX_{12}}{dX_{01}}\ +\ tanh\beta}{1\ +\ \frac{dX_{12}}{dX_{01}}\ tanh\beta} \tag{35}$$

leading to the addition law of areal velocities defined with respect to the temporal bivector coordinate X_{01}

$$V''_{12} = \frac{V'_{12}\ +\ V_{12}}{1\ +\ V'_{12}\ V_{12}}, \quad \frac{dX'_{12}}{dX'_{01}} \equiv V''_{12}, \quad \frac{dX_{12}}{dX_{01}} \equiv V'_{12}, \quad V_{12} \equiv tanh\beta \tag{36}$$

where $V_{12} = tanh\beta$ is the areal velocity of the old frame of reference with respect to the new one. When $\beta = \infty$ one reaches the maximal areal velocity $V_{12} = 1$ in natural units of $c = 1$. If one can restores c the addition law of areal velocities is given by

$$V''_{12} = \frac{V'_{12}\ +\ V_{12}}{1\ +\ \frac{V'_{12}\ V_{12}}{c^2}}, \quad c\ \frac{dX'_{12}}{dX'_{01}} = V''_{12}, \quad c\ \frac{dX_{12}}{dX_{01}} = V'_{12}, \quad V_{12} = c\ tanh\beta \tag{37}$$

leading to the maximal areal velocity (defined with respect to the temporal bivector coordinate X_{01}) given by c, as in ordinary Special Relativity. As a reminder, X_0 has units of length, like ct so one needs to divide by c in order to obtain units of time. X_{01} has units of $(length)^2$ so dividing by c yields units of time × length; etc...

An areal boost transformation defined with respect to the temporal trivector component X_{012}, instead of the ordinary temporal X_0 coordinate and the temporal bivector X_{01} one, is given for example by

$$\begin{aligned} X'_{12} &= X_{12}\ cosh\beta\ + L^{-1}\ X_{012}\ sinh\beta \\ X'_{012} &= X_{012}\ cosh\beta\ + L\ X_{12}\ sinh\beta \\ X'_0 &= X_0, \quad X'_1 = X_1, \quad X'_2 = X_2 \\ X'_{01} &= X_{01}, \quad X'_{02} = X_{02} \end{aligned} \tag{38}$$

Upon doing so the subinterval

$$L^2\ (X'_{12})^2\ -\ (X'_{012})^2\ =\ L^2\ (X_{12})^2\ -\ (X_{012})^2 \tag{39}$$

remains invariant, for the $3D$ spacetime signature $(-,+,+)$, under the transformations (38).

However, if one uses instead the signature $(+,-,-)$ it leads to

$$L^2\ (X'_{12})^2\ +\ (X'_{012})^2\ \neq\ L^2\ (X_{12})^2\ +\ (X_{012})^2 \tag{40}$$

and the latter subinterval (40) is *not* invariant under the boosts provided by eq-(38). We see how transformations in C-space are *signature sensitive*. The reason being that in the combination $L^2(X_{12})^2 + (X_{012})^2$ both the spatial areal components $(X_{12})^2$ and the trivector temporal $(X_{012})^2$ ones appear with the *same* sign. Whereas in the former combination $L^2\ (X_{12})^2\ -\ (X_{012})^2$ the spatial bivector and temporal trivector components appear with *opposite* sign, as they should. Hence we arrive at an important conclusion :

transformations in C-space $favor$ one signature over another. This is not surprising since Clifford algebras distinguish the signatures. The real Clifford algebras $Cl(p,q), Cl(q,p)$ where $p+q=D$ are not isomorphic in general, except in some very special cases.

Now we are going to provide a physical argument as to why the length parameter L admits the *minimal* length scale physical interpretation. The argument relies entirely on the physics behind the extended notion of Lorentz transformations in C-space, and *does not* invoke Quantum Gravity arguments nor quantum group deformations of Lorentz/Poincare algebras.

Fixing the signature $(-,+,+)$, after using the areal boosts transformations of eq-(38) associated with the trivector temporal coordinate X_{012} , and taking similar steps as those provided in eqs-(23,35,36,37), the addition law of areal velocities becomes in this case

$$V''_{12} \ = \ \frac{V'_{12} \ + \ V_{12}}{1 \ + \ \frac{V'_{12}\ V_{12}}{L^{-2}c^2}}, \quad c\,\frac{dX'_{12}}{dX'_{012}} = V''_{12}, \quad c\,\frac{dX_{12}}{dX_{012}} = V'_{12}, \quad V_{12} = L^{-1}c\ tanh\beta \tag{41}$$

Therefore, from the addition law (41) one can infer that if the *maximum* values of the areal velocities $c\,dX_{12}/dX_{012}$ (measured with respect to the temporal trivector coordinates) and the areal velocity of the new frame of reference $V_{12} = L^{-1}c\ tanh\beta$, are *not* infinite but have an upper bound given by c/L, then we must have that L has to be a *minimal* length scale, because c is the *upper* maximum speed in ordinary Special Relativity. Such minimal scale L can be set equal to the Planck scale L_P. As $\beta \rightarrow \infty$ one has that $V_{12} = L^{-1}c\ tanh\beta \rightarrow c/L$, and the addittion/subtraction law (41) when $V'_{12} = V_{12} = c/L$ gives $V''_{12} = c/L$ as expected.

Concluding, from the areal velocity addition law (41) we have shown why the length parameter L needed to be introduced in the C-space interval (2), in order to match physical units, has the physical interpretation of a *minimal* length. The physics of the Extended Relativity theory in C-spaces requires the introduction of the speed of light and a minimal scale. In [2] we have shown how the construction of an Extended Relativity Theory in Clifford *Phase* Spaces requires the introduction of a *maximal* scale which can be identified with the Hubble scale and leads to Modifications of Gravity at the Planck/Hubble scales. Born's Reciprocal Relativity demands that a minimal length corresponds to a minimal momentum that can be set to be $p_{min} = \hbar/R_{Hubble}$. For full details we refer to [2].

One must emphasize that despite the fact that the length parameter L introduced in the C-space interval (2) has the physical interpretation of a *minimal* length, this does *not* mean that the spatial separation between two events in C-space cannot be *smaller* than L. The Planck scale minimal length argument is mainly associated with Quantum Mechanics and Black Hole Physics. The energy involved in the physical measurement process to localize a Planck mass particle, within Planck scale resolutions, becomes very large and such that a black hole forms enclosing the particle behind the black hole horizon. Since one does not have physical access to the black hole interior one cannot probe scales beyond the Planck scale. We shall set aside for the moment the current firewall controversy of black holes.

The theory of Scale Relativity proposed by Nottale [12] is based on a minimal observational length-scale, the Planck scale, as there is in Special Relativity a maximum speed, the speed of light, and deserves to be looked within the Clifford algebraic perspective. In [18] we showed how the minimal length stringy uncertainty relations and further generalizations to include p-brane effects followed from the generalized dispersion relations in C-spaces. The contribution of the polyvector-valued momentum coordinates to the dispersion relations in C-space can be encoded via an effective momentum dependent Planck "constant" $\hbar_{eff}(|\vec{p}|^2)$. Hence there are modifications to the Wely-Heisenberg algebra $[x^i, p_j] = i\hbar_{eff}(|\vec{p}|^2)\delta^i_j$ leading to the stringy uncertainty relations when $\hbar_{eff} = \hbar(1+a|\vec{p}|^2)$. The membrane and p-brane contributions require to have $\hbar_{eff} = \hbar(1+a_1|\vec{p}|^2+a_2|\vec{p}|^4+......)$ stemming from the higher grade polyvector valued momentum components.

We conclude by pointing out that in the quantization program a key role must be played by quantum Clifford-Hopf algebras since the latter q-Clifford algebras naturally contain the κ-deformed Poincare algebras [20], [21], which are essential ingredients in the formulation of DSR. The Minkowski spacetime quantum Clifford algebra structure associated with the conformal group and the Clifford-Hopf alternative κ-deformed quantum Poincare algebra was investigated [19]. The resulting algebra is equivalent to the deformed anti-de Sitter algebra $U_q(so(3,2))$, when the associated Clifford-Hopf algebra is taken into account, together with the associated quantum Clifford algebra and a (not braided) deformation of the periodicity Atiyah-Bott-Shapiro theorem [23].

APPENDIX

In this Appendix we shall write the (anti) commutator relations for the Clifford algebra generators.

$$\frac{1}{2}\{\ \gamma_a,\ \gamma_b\ \} \ = \ g_{ab}\ \mathbf{1};\quad \frac{1}{2}[\ \gamma_a,\ \gamma_b\ ] \ = \ \gamma_{ab} \ = \ -\ \gamma_{ba},\quad a,b = 1,2,3,\cdots,m \qquad (A.1)$$

$$[\ \gamma_a,\ \gamma_{bc}\ ] \ = \ 2\ g_{ab}\ \gamma_c \ - 2\ g_{ac}\ \gamma_b,\quad \{\ \gamma_a,\ \gamma_{bc}\ \} \ = \ 2\ \gamma_{abc} \qquad (A.2)$$

$$[\ \gamma_{ab},\ \gamma_{cd}\ ] \ = \ -2\ g_{ac}\ \gamma_{bd}\ + 2\ g_{ad}\ \gamma_{bc}\ - 2\ g_{bd}\ \gamma_{ac}\ + 2\ g_{bc}\ \gamma_{ad} \qquad (A.3)$$

In general one has [11]

$$pq = \mathbf{odd},\ \ [\gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}] \ = \ 2\ \gamma^{n_1n_2....n_q}_{m_1m_2....m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[n_1n_2}_{[m_1m_2}\ \gamma^{n_3....n_q]}_{m_3......m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[n_1....n_4}_{[m_1....m_4}\ \gamma^{n_5....n_q]}_{m_5......m_p]}\ -\ .... \qquad (A.4)$$

$$pq = \mathbf{even},\ \ \{\ \gamma_{m_1m_2....m_p},\ \gamma^{n_1n_2....n_q}\ \} \ = \ 2\ \gamma^{n_1n_2....n_q}_{m_1m_2....m_p} - \frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[n_1n_2}_{[m_1m_2}\ \gamma^{n_3....n_q]}_{m_3......m_p]} +$$

$$\frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[n_1....n_4}_{[m_1....m_4}\ \gamma^{n_5....n_q]}_{m_5......m_p]}\ -\ .... \qquad (A.5)$$

$$pq = \mathbf{even}, \quad [\gamma_{m_1 m_2 ....m_p}, \ \gamma^{n_1 n_2 ....n_q}] \ = \ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[n_1}_{[m_1} \ \gamma^{n_2....n_q]}_{m_2....m_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[n_1 n_2 n_3}_{[m_1 m_2 m_3} \ \gamma^{n_4....n_q]}_{m_4......m_p]} \ + \ .... \qquad (A.6)$$

$$pq = \mathbf{odd}, \quad \{ \ \gamma_{m_1 m_2 ....m_p}, \ \gamma^{n_1 n_2 ....n_q} \ \} \ = \ \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta^{[n_1}_{[m_1} \ \gamma^{n_2....n_q]}_{m_2....m_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[n_1 n_2 n_3}_{[m_1 m_2 m_3} \ \gamma^{n_4....n_q]}_{m_4......m_p]} \ + \ .... \qquad (A.7)$$

The generalized Kronecker delta is defined as the determinant

$$\delta^{a_1 a_2 .....a_k}_{b_1 b_2 .....b_k} \ \equiv \ det \begin{pmatrix} \delta^{a_1}_{b_1} & \ldots & \ldots & \delta^{a_1}_{b_k} \\ \delta^{a_2}_{b_1} & \ldots & \ldots & \delta^{a_2}_{b_k} \\ ------ & ------ & ------ & ------ \\ \delta^{a_k}_{b_1} & \ldots & \ldots & \delta^{a_k}_{b_k} \end{pmatrix} \qquad (A.8)$$

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