Euler's Equation and Quaternions

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Summary

The objective of this text is to present a method of using vectors and quaternions to produce Euler's Equation. The method presented uses the cross product of vectors in the **j**-**k** plane to produce the **i**sine portion of Euler's Equation. The cosine portion of Euler's Equation is produced by the dot product of the same vectors. The method is then generalized to apply to quaternions.

Preface

A knowledge of vectors and quaternions is required.

Discussion

This text will present a method to combine the vector dot product with the vector cross product to produce Euler's Equation. Despite the extreme simplicity of what is presented, the author was unable to find this method presented on the internet or in any textbook available to the author, including the texts produced by Hamilton on the subject of quaternions.

The first step is to state Euler's Equation.

Euler's Equation[1]:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Here, the i term is normally thought of as the complex i rather than as a vector.

The next step is to state a suitable definition for the dot product and cross product of two vectors. A lower-case, bold-faced letter such as "**a**" is used to designate a vector.

Dot Product[2]:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$

Cross Product[3]:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta)\,\mathbf{n}$$

The geometric meaning of the dot product is the length of the projection of one of the vectors onto the other. For the cross product, the vector **n** is a unit vector that is normal (perpendicular) to both vector **a** and vector **b**. The geometric interpretation of the cross product is then a vector perpendicular to the plane of vectors **a** and **b**. It has a vector length equal to the area associated with the parallelogram produced by vectors **a** and **b** (i.e., not simply the triangle produced by **a** and **b**).

Vectors **a** and **b** can be represented in general as follows:

$$\mathbf{a} = a_i\mathbf{i} + a_j\mathbf{j} + a_k\mathbf{k}$$

$$\mathbf{b} = b_i\mathbf{i} + b_j\mathbf{j} + b_k\mathbf{k}$$

If vectors **a** and **b** are both restricted to the **j**-**k** plane (i.e., $a_i = b_i = 0$), then their cross product will be in the **i** direction. This is the essence of the method presented here. Euler's Equation can now be written as follows:

Euler's Equation (vector form):

$$e^{\mathbf{i}\theta} = \frac{1}{|\mathbf{a}||\mathbf{b}|}(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b});\ a_i = b_i = 0$$

In this form, the complex i has become the vector **i**. In the author's opinion, the most amazing things about this form of Euler's Equation are that the **i**θ term has become hidden on the right-hand side and that any arbitrary, non-collinear vectors **a** and **b** will satisfy it, provided they are restricted to the **j**-**k** plane.

The author will now extend this concept to quaternions. Let us extend vectors **a** and **b** to become quaternions **A** and **B** as follows (remembering that **a** and **b** are restricted to the **j**-**k** plane):

$$\mathbf{A} = a_0 + a_j\mathbf{j} + a_k\mathbf{k} = a_0 + \mathbf{a}$$

$$\mathbf{B} = b_0 + b_j\mathbf{j} + b_k\mathbf{k} = b_0 + \mathbf{b}$$

One of the identities[4] of quaternions is as follows:

$$\mathbf{AB} = a_0 b_0 - \mathbf{a}\cdot\mathbf{b} + a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a}\times\mathbf{b}$$

If both sides of this identity are divided by the product of the lengths of vectors **a** and **b**, the result is:

$$\frac{1}{|\mathbf{a}||\mathbf{b}|}\mathbf{AB} = \frac{1}{|\mathbf{a}||\mathbf{b}|}(a_0 b_0 - \mathbf{a}\cdot\mathbf{b} + a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a}\times\mathbf{b})$$

Now, perform a little algebra to make this similar to Euler's Equation.

$$\frac{1}{|\mathbf{a}||\mathbf{b}|}\mathbf{AB} = \frac{1}{|\mathbf{a}||\mathbf{b}|}(a_0 b_0 - 2\mathbf{a}\cdot\mathbf{b} + a_0\mathbf{b} + b_0\mathbf{a}) + \frac{1}{|\mathbf{a}||\mathbf{b}|}(\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\times\mathbf{b})$$

Now, substitute in the complex exponential form of Euler's Equation.

$$\frac{1}{|\mathbf{a}||\mathbf{b}|}\mathbf{AB} = \frac{1}{|\mathbf{a}||\mathbf{b}|}(a_0 b_0 - 2\mathbf{a}\cdot\mathbf{b} + a_0\mathbf{b} + b_0\mathbf{a}) + e^{\mathbf{i}\theta};\ a_i = b_i = 0$$

Now, rearrange slightly.

Euler's Equation (quaternion form):

$$e^{\mathbf{i}\theta} = \frac{1}{|\mathbf{a}||\mathbf{b}|}[\mathbf{AB} - (a_0 b_0 - 2\mathbf{a}\cdot\mathbf{b} + a_0\mathbf{b} + b_0\mathbf{a})]\ ;\ a_i = b_i = 0$$

There are similar identities for complex exponentials based upon **j** and **k**.

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References

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