Some Observations on Schrödinger's Affine Connection

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February 22, 2014

Abstract

In a series of papers written over the period 1944-1948, the great Austrian physicist Erwin Schrödinger presented his ideas on symmetric and non-symmetric affine connections and their possible application to general relativity. Several of these ideas were subsequently presented in his notable 1950 book *Space-Time Structure*, in which Schrödinger outlined the case for both metric and general connections, symmetric and otherwise. In the following discussion we focus on one particular connection presented by Schrödinger in that book and its relationship with the non-metricity tensor $D_\lambda g_{\mu\nu}$. We also discuss how this connection overcomes a problem that Hermann Weyl experienced with the connection he proposed in his failed 1918 theory of the combined gravitational-electromagnetic field. A simple physical argument is then presented demonstrating that Schrödingers's formalism accommodates electromagnetism in a more natural way than Weyl's theory.

1. Introduction

In early 1918 the German mathematician Hermann Weyl proposed a generalization of Riemannian geometry in an effort to unify the two forces of Nature then known, gravitation and electromagnetism. Weyl developed his theory by relaxing one of the tenets of Riemannian geometry, the invariance of vector magnitude or length under parallel transport. This required that the covariant derivative of the metric tensor $g_{\mu\nu}$ not vanish, which led to Weyl's introduction of a new vector quantity that he identified as the electromagnetic four-vector A_μ. The theory failed, but it produced a non-Riemannian geometry that is still of considerable interest today.

The main problem with Weyl's geometry can be traced to its inability to accommodate fixed-length vectors under parallel transfer. As is well known, Einstein seized upon this aspect of the theory to argue that certain physical quantities would vary arbitrarily under a local scale of scale, with the result that phenomena such as the spacing of atomic spectral lines would depend on their history. A related problem concerned the fact that the lengths of vectors such as the four-momentum p^μ and even the unit vector dx^μ/ds would vary in time and space — a clearly non-physical presumption.

2. Notation

Following Adler et al., we denote ordinary partial differentiation with a single subscripted bar, while covariant differentiation is denoted using double subscripted bars. Schrödinger's connection is representated by the symbol Γ, while its representation in Riemannian space is the usual Christoffel bracket. Thus, the covariant derivative of the mixed tensor $F^\lambda_{\ \alpha}$ is given as

$$F^\lambda_{\ \alpha||\beta} = F^\lambda_{\ \alpha|\beta} - F^\lambda_{\ \mu}\Gamma^\mu_{\alpha\beta} + F^\mu_{\ \alpha}\Gamma^\lambda_{\mu\beta}$$

while for Riemannian space we make the substitution

$$\Gamma^\mu_{\alpha\beta} \to \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}$$

where

$$\left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu|\beta} + g_{\nu\beta|\alpha} - g_{\alpha\beta|\nu} \right)$$

The metric tensor $g_{\mu\nu}$ is taken to be symmetric in its lower indices.

3. The connection idea

The *connection* in differential geometry is normally introduced using either the notion of a *constant vector* whose components are fixed in some given coordinate system, or an arbitrary vector that is unchanged in direction under parallel transport. Either of these approaches can be used to define the familiar process known as covariant differentiation. Historically, Cartan and Weyl pioneered the concept of parallel transfer by assuming that the infinitesimal change in an arbitrary vector ξ^μ under physical transport is proportional to the transport interval dx and the initial vector at point x,

$$\delta\xi^\mu = -\Gamma^\mu_{\alpha\beta}\,\xi^\alpha dx^\beta, \tag{3.1}$$

such that the quantity $\xi^\mu + \delta\xi^\mu$ represents the "unchanged" or parallel vector at the point $x + dx$. Noting that the vector will physically change by the amount $d\xi^\mu$ after transport, where

$$d\xi^\mu = \frac{\partial\xi^\mu}{\partial x^\lambda} dx^\lambda,$$

we can then define the covariant derivative of the vector as

$$\xi^\mu_{\;||\lambda} = \lim_{dx^\lambda\to 0} \frac{(\xi^\mu + d\xi^\mu) - (\xi^\mu + \delta\xi^\mu)}{dx^\lambda}$$

or

$$\xi^\mu_{\;||\lambda} = \xi^\mu_{\;|\lambda} + \Gamma^\mu_{\alpha\lambda}\xi^\alpha \tag{3.2}$$

Note in the above that the connection Γ is, to a considerable extent, completely arbitrary. Assuming no particular symmetry of the indices, there are $4^3 = 64$ independent components in the connection term in four dimensions.

One immediate consequence of this formalism is the notion of a *geodesic*. Consider the case where the vector in question is the unit vector dx^μ/ds. We might properly assume that this vector should be unchanged under parallel transfer, which means that its covariant derivative vanishes:

$$\left(\frac{dx^\mu}{ds}\right)_{|\lambda} + \;\Gamma^\mu_{\alpha\lambda}\frac{dx^\alpha}{ds} = 0$$

Multiplying by dx^λ/ds and summing, we have

$$\left(\frac{dx^\mu}{ds}\right)_{|\lambda}\frac{dx^\lambda}{ds} + \;\Gamma^\mu_{\alpha\lambda}\frac{dx^\alpha}{ds}\frac{dx^\lambda}{ds} = 0$$

or

$$\frac{d^2x^\mu}{ds^2} + \;\Gamma^\mu_{\alpha\lambda}\frac{dx^\alpha}{ds}\frac{dx^\lambda}{ds} = 0$$

Here we see that the issue of symmetry in the covariant indices of the connection is irrelevant, since any antisymmetry is canceled out in the second term. For a symmetric connection in four dimensions, the number of arbitrary, independent components is then reduced to $4\cdot 4\cdot(4+1)/2 = 40$.

4. Metricity

Riemannian geometry is characterized by a condition known as *metricity*, meaning that the covariant derivative of the metric tensor $g_{\mu\nu}$ vanishes:

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha} = 0$$

Metricity thus requires that the lower indices of the connection be symmetric. By an appropriate permutation of the above expression, it also provides a means for uniquely identifying all 40 components of the connection in terms of the metric tensor and its first derivatives:

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\alpha\beta}\left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta}\right)$$

Metricity is usually assumed on the basis of either simplicity or physical validity (Einstein's general relativity theory does work, after all) or by demanding that all quantities in an invariant product be parallel-transfer invariant themselves. For example, the magnitude of the unit vector, given by

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1$$

represents such a product and, since the unit vector itself is transfer-invariant, we might assume the same to be true for the metric tensor as well.

5. Weyl's theory as an example of non-metricity

The non-vanishing of the metric covariant derivative is called *non-metricity*, and $g_{\mu\nu||\alpha}$ is called the non-metricity tensor. Over the years it has been investigated by legions of physicists searching for an alternative to Riemannian geometry, usually in the guise of unified field theory. To date none of these investigations has been successful.

In his 1918 theory, Weyl abandoned the notion of invariance of vector length, and was thus led to a geometry involving non-metricity. Weyl also rejected the assumption that the metric tensor parallel-transferred to itself and, with vector length defined by $L^2 = g_{\mu\nu}\xi^\mu\xi^\nu$, he arrived at

$$2L\,dL = g_{\mu\nu|\alpha}\xi^\mu\xi^\nu dx^\alpha dx^\alpha - g_{\mu\nu}\xi^\mu\delta\xi^\nu - g_{\mu\nu}\xi^\nu\delta\xi^\mu$$

Using the transport law (3.1) and relabeling indices, this becomes

$$\begin{aligned} 2L\,dL &= \left(g_{\mu\nu|\alpha} - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha}\right)\xi^\mu\xi^\nu dx^\alpha \\ &= g_{\mu\nu||\alpha}\xi^\mu\xi^\nu dx^\alpha \end{aligned} \tag{5.1}$$

Since Weyl had no idea what comprised the connection term, he reasonably surmised that, like the vector transport law, the differential change in length dL was also linear with respect to the vector and the distance. He thus wrote

$$dL = L\,\phi_\mu dx^\mu \tag{5.2}$$

where ϕ_μ was some as yet undefined vector quantity. With the use of (5.2) Weyl was able to express the non-metricity tensor as

$$g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha \tag{5.3}$$

He also noted that the resulting expression for the change in length $dL = L\phi_\mu dx^\mu$ could be immediately integrated to give

$$L = L_0\, e^{\int \phi_\mu dx^\mu} \tag{5.4}$$

where L_0 is the initial vector length. Thus, in Weyl's geometry the length of a vector depends inherently on the vector field ϕ_μ, which Weyl subsequently identified as the four-potential A_μ of electromagnetism.

Weyl's assumed identity for the non-metricity tensor (5.3) allowed him to uniquely identify the components of his connection. By taking cyclic permutations of the expanded form

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha}$$

Weyl was able to show that

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \delta^\alpha_\nu\phi_\mu - \delta^\alpha_\mu\phi_\nu + g_{\mu\nu}g^{\alpha\beta}\phi_\beta \tag{5.5}$$

Historically, it was (5.4) that Einstein had difficulty with, for no matter how small the Weyl field ϕ_μ is taken the length of a vector would change continuously and arbitrarily from place to place and from time to time. For example, the free-space momentum and Compton wavelength vectors

$$p^\mu = mc\frac{dx^\mu}{ds},$$
$$p^\mu = \frac{mc}{\hbar}\frac{dx^\mu}{ds}$$

would vary in Weyl's theory, in contradiction to both reason and observation. In fact, the length of *any* vector quantity would undergo a change in magnitude in Weyl's theory, the simplest example being the unit tangent vector dx^α/ds itself. Given the invariant line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, the length of the unit vector dx^μ/ds would thus change in the presence of Weyl's ϕ-field — a nonsensical prediction.

6. Schrödinger's affine connection

In his early work on connections Schrödinger assumed metricity, but he also considered non-symmetric connections as well. In his book *Space-Time Structure* he uses cyclic permutations of the metricity condition

$$g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - g_{\lambda\nu}\Gamma^\lambda_{\mu\alpha} = 0$$

to show that

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta}\left(g_{\lambda\nu}\Gamma^\lambda_{\mu\beta-} + g_{\mu\lambda}\Gamma^\lambda_{\nu\beta-} \right) + \Gamma^\alpha_{\mu\nu-}$$

where

$$\Gamma^\alpha_{\lambda\beta-} = \frac{1}{2}\left(\Gamma^\alpha_{\lambda\beta} - \Gamma^\alpha_{\beta\lambda}\right), \quad \Gamma^\alpha_{\lambda\beta+} = \frac{1}{2}\left(\Gamma^\alpha_{\lambda\beta} + \Gamma^\alpha_{\beta\lambda}\right), \quad \Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu+} + \Gamma^\alpha_{\mu\nu-}$$

Noting that the first two terms on the right are symmetric and that the equations of the geodesics (3.2) cancel any antisymmetry in the overall connection, Schrödinger at this point decides to simply omit the last term, producing a purely symmetric connection composed of the metric tensor and its first derivatives and two skew terms that together comprise a new *tensor* of rank three, which Schrödinger calls $T_{\beta\mu\nu}$. He thus writes

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta} T_{\beta\mu\nu} \tag{6.1}$$

Schrödinger notes that while the T-tensor is completely arbitrary, it has interesting symmetry properties:

$$\begin{aligned} T_{\beta\mu\nu} = T_{\beta\nu\mu} \quad &\text{and} \\ T_{\beta\mu\nu} + T_{\nu\beta\mu} + T_{\mu\nu\beta} &= 0 \end{aligned} \tag{6.2}$$

Careful consideration of these symmetry properties shows that the number of independent components for $T_{\beta\mu\nu}$ in n dimensions is $n(n^2-1)/3$, or 20 components in four dimensions.

In his book Schrödinger asserts that (6.1) represents the widest possible class of connections compatible with an arbitrary given $g_{\mu\nu}$. More importantly it is probably the simplest connection that can be written, with the exception of the Christoffel symbol itself.

7. Non-metricity and Schrödinger's connection

Although Schrödinger assumed metricity of the geometry from the beginning, it can be shown that this assumption is not necessary. Returning to our derivation for the change in vector magnitude (5.1), we note that Weyl's presumed identity for the non-metricity tensor $g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha$ is the reason why there cannot be any fixed-length vectors in the Weyl theory. If we repeat the same analysis for the unit vector dx^μ/ds, we have

$$2LdL = g_{\mu\nu||\alpha}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}dx^\alpha$$

But this must be identically zero if fixed-length vectors such as this are to be transfer-invariant. Thus,

$$g_{\mu\nu||\alpha}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}dx^\alpha = 0$$

or, equivalently,

$$g_{\mu\nu||\alpha}dx^\mu dx^\nu dx^\alpha = 0$$

This indicates that either the non-metricity tensor vanishes identically or it satisfies the cyclic symmetry condition

$$g_{\mu\nu||\alpha} + g_{\alpha\mu||\nu} + g_{\nu\alpha||\mu} = 0 \tag{7.1}$$

We can now show that this condition in fact leads to Schrödinger's connection. If we subtract

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha} - g_{\lambda\nu}\Gamma^{\lambda}_{\mu\alpha}$$

from its two cyclic permutations, we have

$$g_{\mu\nu||\alpha} + g_{\alpha\mu||\nu} - g_{\nu\alpha||\mu} = 2g_{\alpha\lambda}\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - 2g_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu}$$

or, in view of (7.1),

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta}g_{\mu\nu||\beta} \tag{7.2}$$

Comparing this with (6.1), we see that Schrödinger's T-tensor and the non-metricity tensor have identical symmetry properties. In view of the naturalness of its derivation, it seems completely plausible to conclude that the T-tensor and $g_{\mu\nu||\beta}$ are the same quantity. Thus, the most general connection sought by Schrödinger is (7.2). More importantly, this connection automatically accommodates fixed-length vectors in the formalism.

8. Schrödinger's connection and electromagnetism

Weyl was able to relate gravitation and electromagnetism through a vector field ϕ_μ that was responsible for the non-metricity of his theory. Under a scale (or *conformal*) transformation of the metric tensor $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$, Weyl determined that his vector field simultaneously transformed according to $\phi_\mu \to \phi_\mu + \log\Omega_{|\mu}$. The similarity of this transformation with the familiar gauge transformation property of the electromagnetic four-potential A_μ led Weyl to believe he had unified gravitation with electromagnetism, particularly since his connection was fully invariant to a change in the metric scale.

By comparison, Schrödinger's connection is not scale invariant (although it can be shown that the contracted form $\Gamma^{\mu}_{\alpha\mu}$ is invariant), and there seems to be little to be gained by trying to make a conformal theory of gravity from it, much less tie it to electromagnetism. Consequently, Schrödinger's connection should not be seen as "unifying" anything. However, it is interesting to note that Schrödinger's Γ appears to at least *accommodate* electromagnetism, which we will now demonstrate with the following simple exercise.

We consider the geodesic equations again,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

Using Schrödinger's connection, we can write this as

$$\frac{d^2x^\alpha}{ds^2} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -g^{\alpha\beta}g_{\mu\nu||\beta}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

The non-metricity term on the right-hand side would appear to be a Lorentz force term f^α. For a particle of mass m and charge q, the Lorentz force is known to be

$$f^\alpha = \frac{q}{mc}F^{\alpha}_{\ \beta}\frac{dx^\beta}{ds}$$

This implies that

$$\frac{q}{mc}F^{\alpha}_{\ \beta}\frac{dx^\beta}{ds} = -g^{\alpha\beta}g_{\mu\nu||\beta}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds},$$

or, equivalently,

$$\frac{q}{mc}F_{\alpha\beta}\frac{dx^\beta}{ds} = -g_{\mu\nu||\alpha}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

This can be considerably simplified using several convenient identities. Using (7.1), we see that

$$g_{\mu\nu||\alpha}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -2g_{\mu\alpha||\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}, \tag{8.1}$$

while from the equations of the geodesics themselves we have the simple and useful identity

$$\left(\frac{dx^{\mu}}{ds}\right)_{||\nu}\frac{dx^{\nu}}{ds}=0 \tag{8.2}$$

Another is obtained via covariant differentiation of the length of the unit vector

$$1=g_{\mu\beta}\frac{dx^{\mu}}{ds}\frac{dx^{\beta}}{ds}$$

Differentiation with respect to x^{α} gives

$$0=g_{\mu\beta||\alpha}\frac{dx^{\mu}}{ds}\frac{dx^{\beta}}{ds}+2\,g_{\mu\beta}\left(\frac{dx^{\mu}}{ds}\right)_{||\alpha}\frac{dx^{\beta}}{ds}$$

or

$$g_{\mu\beta}\left(\frac{dx^{\nu}}{ds}\right)_{||\alpha}\frac{dx^{\beta}}{ds}=-\frac{1}{2}g_{\mu\beta||\alpha}\frac{dx^{\mu}}{ds}\frac{dx^{\beta}}{ds} \tag{8.3}$$

Using (8.1) and (8.2), we can then write

$$\frac{q}{mc}F_{\alpha\beta}\frac{dx^{\beta}}{ds}=2\left(g_{\mu\alpha}\frac{dx^{\mu}}{ds}\right)_{||\beta}\frac{dx^{\beta}}{ds} \tag{8.4}$$

We cannot cancel the common unit vector dx^{β}/ds from both sides of this expression, as we expect $F_{\alpha\beta}$ to be antisymmetric in its indices. However, it is easy to show that, with the use of (8.3),

$$\begin{aligned}\frac{q}{mc}F_{\alpha\beta} &= \left(g_{\mu\alpha}\frac{dx^{\mu}}{ds}\right)_{||\beta}-\left(g_{\mu\beta}\frac{dx^{\mu}}{ds}\right)_{||\alpha}\\ &= \left(g_{\mu\alpha}\frac{dx^{\mu}}{ds}\right)_{|\beta}-\left(g_{\mu\beta}\frac{dx^{\mu}}{ds}\right)_{|\alpha}\end{aligned} \tag{8.5}$$

which reduces to (8.4) when multiplied by dx^{β}/ds. Using the symmetry properties of the Riemann-Christoffel tensor $R^{\mu}_{\alpha\beta\lambda}$, it is a straightforward exercise to show that this definition also satisfies the homogeneous Maxwell equations

$$F_{\alpha\beta||\lambda}+F_{\lambda\alpha||\beta}+F_{\beta\lambda||\alpha}=0$$

The similarity of (8.5) to the familiar $F_{\alpha\beta}=A_{\alpha|\beta}-A_{\beta|\alpha}$ is evident. If we set

$$A_{\alpha}=\frac{q}{mc}\,g_{\mu\alpha}\,\frac{dx^{\mu}}{ds},$$

then it would appear that the Schrödinger connection does indeed accommodate electromagnetism. But again, it is emphasized that this by no means represents a unification of geometry and electromagnetism, since the tensor $F_{\alpha\beta}$ had to be introduced into the formalism from the outside.

9. Last remarks

Researchers have investigated many types of affine connection over the years, the most notable beginning with Weyl in 1918 and Eddington in 1921. A surprising number of these efforts resulted in connections that today seem overly complicated and even bizarre. Einstein himself explored connections for many years, and his final effort to produce a consistent unified field theory in 1954 consisted of a non-symmetric variant. Schrödinger appears to have joined the game fairly late in his career, and in the 1940s he examined many connection forms, along with non-symmetric variants of the metric tensor itself.

Perhaps because of its simplicity, Schrödinger's connection as presented in *Space-Time Structure* seems to have been overlooked. It is hoped that this elementary re-examination of his efforts to find the fundamental connection of the world will rekindle renewed interest in this lesser-known work of the Father of Wave Mechanics.

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