Theoretical derivation of spin

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In four-dimensional or higher-dimensional space, spin angular momentum is not known well. For instance, we are not sure how many components it has in four-dimensional space. In this paper, the spin properties are theoretically derived and investigated in general-dimensional space. Before studying spin, we review the orbital angular momentum properties first, because they are theoretically well known even in higher-dimensional space. Then, for the orbital angular momentum, we make a special but natural assumption, and see that it leads a quantity similar to spin or Pauli matrices. It is remarkable that all the spin properties are derived from the one and only one assumption. Consistency of our consideration is checked by calculating the four-dimensional version of spin spherical harmonics and its eigenvalues. This theory may enable us to understand the double degeneracy of spin-1/2 particles without bringing in something called spin.

I. REVIEW OF ORBITAL ANGULAR MOMENTUM

We summarize the orbital angular momentum properties in general-dimensional space before considering spin.

In n-dimensional space (hereafter n always denotes the dimension of the space), orbital angular momentum $\boldsymbol{L}$ is defined by the wedge product of position $\boldsymbol{x}$ and momentum $\boldsymbol{p}$:

$$\boldsymbol{L} = \boldsymbol{x} \wedge \boldsymbol{p}. \tag{1}$$

By the standard basis of Euclidean space, 2-vector $\boldsymbol{L}$ can be expanded as

$$\boldsymbol{L} = \left(\sum_{i=1}^{n} x_i \boldsymbol{e}_i\right) \wedge \left(\sum_{j=1}^{n} p_j \boldsymbol{e}_j\right) = \sum_{i,j=1}^{n} x_i p_j \boldsymbol{e}_i \wedge \boldsymbol{e}_j, \tag{2}$$

and using anticommutativity $\boldsymbol{e}_i \wedge \boldsymbol{e}_j = -\boldsymbol{e}_j \wedge \boldsymbol{e}_i$, it is rewritten as

$$\boldsymbol{L} = \frac{1}{2} \sum_{i,j=1}^{n} L_{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j \tag{3}$$

with

$$L_{ij} = x_i p_j - x_j p_i. \tag{4}$$

In n-dimensional space, the number of independent components of $\boldsymbol{L}$ is $n(n-1)/2$ because of the antisymmetricity of L_{ij}:

$$L_{ij} = -L_{ji}. \tag{5}$$

The inner product of two general 2-vectors $\boldsymbol{A} = \sum_{i,j} A_{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j$, $\boldsymbol{B} = \sum_{i,j} B_{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j$ is a scalar, and if an orthonormal basis is chosen, it can be calculated as

$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A} = \sum_{i,j=1}^{n} (A_{ij}B_{ij} - A_{ij}B_{ji}). \tag{6}$$

Note the commutativity of the inner product. The inner product of $\boldsymbol{L}$ with itself is given by

$$\boldsymbol{L}^2 = \boldsymbol{L}\cdot\boldsymbol{L} = \frac{1}{2} \sum_{i,j=1}^{n} L_{ij}^2. \tag{7}$$

In quantum mechanics, $\boldsymbol{x}$ and $\boldsymbol{p}$ become operators, and $\boldsymbol{L}$ also becomes an operator, for example, it is represented as

$$L_{ij} = -i\hbar\left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right). \tag{8}$$

From canonical commutation relation

$$[x_i, p_j] = i\hbar\, \delta_{ij}, \tag{9}$$

we obtain the commutation relation between arbitrary two components of $\boldsymbol{L}$:

$$[L_{ij}, L_{k\ell}] = i\hbar\,(\delta_{ik}L_{j\ell} + \delta_{i\ell}L_{kj} + \delta_{jk}L_{\ell i} + \delta_{j\ell}L_{ik}). \tag{10}$$

Using this commutation relation, we can show the commutativity of L_{ij} and $\boldsymbol{L}^2$:

$$[L_{ij}, \boldsymbol{L}^2] = 0. \tag{11}$$

We can also see that L_{ij} satisfies the following identical equation:

$$L_{ij}L_{k\ell} - L_{ik}L_{j\ell} + L_{i\ell}L_{jk} = -i\hbar\,(\delta_{ij}L_{k\ell} - \delta_{ik}L_{j\ell} + \delta_{i\ell}L_{jk}). \tag{12}$$

The eigenvalue problem of $\boldsymbol{L}^2$ can be solved analytically. The eigenvalues are given by

$$\hbar^2\ell(\ell + n - 2), \qquad \ell = 0, 1, 2, \ldots. \tag{13}$$

II. DERIVATION OF SPIN

In Eq. (13), the values of ℓ are set to non-negative integers, but there is no necessity to do so. For instance, in three-dimensional space ($n = 3$), when an eigenvalue of $\boldsymbol{L}^2$ is provided by $\ell(\ell + 1)\hbar^2 = 6\hbar^2$, we can adopt $\ell = -3$ as well as $\ell = 2$. Both non-negative and negative integers are acceptable as values of ℓ. However, in practical problems, we do not need to mind if ℓ is negative or non-negative, because information about the sign of ℓ never appears in the eigenvalues and eigenfunctions of $\boldsymbol{L}^2$. The theory of $\boldsymbol{L}^2$ has the complete symmetry with respect to the interchange of ℓ and $-(\ell + n - 2)$. In Eq. (13), we just take non-negative ℓ for convenience and this is the usual manner.

Nevertheless, in this paper, we would really like to distinguish between non-negative ℓ and negative $-(\ell + n - 2)$. To this end, we presume that there exists a new operator which simultaneously has both the eigenvalues, $\hbar\ell$ and $-\hbar(\ell + n - 2)$. This assumption implies some kind of symmetry breaking between ℓ and $-(\ell + n - 2)$, but it is known that this symmetry is never broken as long as we see the *square* operator $\boldsymbol{L}^2$. Thus we suppose that there exists a *not squared* orbital angular momentum operator. The beingness of the not squared operator is the only and essential postulate of this theory.

Let us seek the not squared orbital angular momentum operator. The only clue we know is the eigenvalues of the operator:

$$\hbar\ell,\ -\hbar(\ell + n - 2), \qquad \ell = 0, 1, 2, \ldots. \tag{14}$$

These eigenvalues have the dimension equivalent to the angular momentum. Note that the operator we are looking for is not $\boldsymbol{L}$ itself, because $\boldsymbol{L}$ is a 2-vector operator and its eigenvalues must be 2-vector also. Scalar eigenvalues (14) require an operator with scalar form. Therefore, using the inner product of 2-vectors, we assume the form of the operator as

$$\boldsymbol{\sigma}\cdot\boldsymbol{L} \tag{15}$$

with

$$\boldsymbol{\sigma} = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij}\boldsymbol{e}_i \wedge \boldsymbol{e}_j, \tag{16}$$

where $\boldsymbol{\sigma}$ is an undetermined 2-vector operator. In general, the components of $\boldsymbol{\sigma}$ do not need to have the antisymmetricity

$$\sigma_{ij} = -\sigma_{ji}, \tag{17}$$

but using the anticommutativity of wedge product, we can always make $\boldsymbol{\sigma}$ to be antisymmetric. In fact, if we transform Eq. (16) as

$$\boldsymbol{\sigma} = \frac{1}{2}\sum_{i,j=1}^{n} \frac{1}{2}(\sigma_{ij} - \sigma_{ji})\, \boldsymbol{e}_i \wedge \boldsymbol{e}_j \tag{18}$$

and regard $(\sigma_{ij}-\sigma_{ji})/2$ as a new component of $\boldsymbol{\sigma}$, the obtained new components have the antisymmetricity. Hereafter we always assume the antisymmetricity of σ_{ij}. Using Eq. (6), we rewrite Eq. (15) as

$$\boldsymbol{\sigma}\cdot\boldsymbol{L} = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij} L_{ij}. \tag{19}$$

In the rest of this section, we fix the form of the not squared operator as Eq. (19) and study the properties of the undetermined coefficient $\boldsymbol{\sigma}$.

From Eq. (14), it is obvious that the eigenvalues of $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ are real numbers. This means that $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ is Hermitian:

$$\boldsymbol{\sigma}\cdot\boldsymbol{L} = (\boldsymbol{\sigma}\cdot\boldsymbol{L})^{\dagger}. \tag{20}$$

The commutativity of the inner product requires

$$\boldsymbol{\sigma}\cdot\boldsymbol{L} = \boldsymbol{L}\cdot\boldsymbol{\sigma}, \tag{21}$$

and Eq. (20) yields

$$\boldsymbol{\sigma}\cdot\boldsymbol{L} = \boldsymbol{\sigma}^{\dagger}\cdot\boldsymbol{L}. \tag{22}$$

To make this relation hold, we just take $\boldsymbol{\sigma}$ as a Hermitian operator:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\dagger}. \tag{23}$$

This is the first property of $\boldsymbol{\sigma}$.

Practical measurement and observation of $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ for a certain eigenstate permits us to know the eigenvalue of $\boldsymbol{L}^2$ for the same state. For instance, in three-dimensional space, when eigenvalue $-3\hbar$ is obtained by measurement of $\boldsymbol{\sigma}\cdot\boldsymbol{L}$, we will always obtain eigenvalue $6\hbar^2$ in the succeeding $\boldsymbol{L}^2$ measurement for the same state. This indicates that $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ and $\boldsymbol{L}^2$ can be observed simultaneously and it is mathematically written as

$$[\boldsymbol{\sigma}\cdot\boldsymbol{L}, \boldsymbol{L}^2] = 0. \tag{24}$$

In order to make this commutation relation hold, we require more fundamental or strong commutation relation:

$$[\sigma_{ij}, L_{k\ell}] = 0, \qquad i,j,k,\ell = 1,\ldots,n. \tag{25}$$

This commutation relation suggests that σ_{ij} is like a constant; in other words, σ_{ij} does not contain position x_k or differential operator $\partial/\partial x_k$ (see Eq. (8)). This constant assumption is not so artificial because $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ is thought to be a scalar version of $\boldsymbol{L}$, and σ_{ij} seems to be corresponding to $\boldsymbol{e}_i \wedge \boldsymbol{e}_j$ in Eq. (3), which is also like a constant in Euclidean space (compare the right-hand sides of Eq. (3) and Eq. (19)). To prove Eq. (25) yielding Eq. (24), we expand the left-hand side of Eq. (24) as

$$[\boldsymbol{\sigma}\cdot\boldsymbol{L}, \boldsymbol{L}^2] = \frac{1}{2}\sum_{i,j=1}^{n}\big(\sigma_{ij}[L_{ij}, \boldsymbol{L}^2] + [\sigma_{ij}, \boldsymbol{L}^2]L_{ij}\big). \tag{26}$$

From Eqs. (11) and (25), this expression immediately becomes zero. The commutation relation (25) is the second property of $\boldsymbol{\sigma}$.

Lastly, let us see the most important relation which characterizes $\boldsymbol{\sigma}$. Two identities

$$\begin{aligned} (\hbar\ell)^2 + \hbar(n-2)(\hbar\ell) &= \hbar^2\ell(\ell+n-2), \\ \big(-\hbar(\ell+n-2)\big)^2 + \hbar(n-2)\big(-\hbar(\ell+n-2)\big) &= \hbar^2\ell(\ell+n-2) \end{aligned} \tag{27}$$

imply that the following operator identity holds:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})^2 + \hbar(n-2)\,\boldsymbol{\sigma}\cdot\boldsymbol{L} = \boldsymbol{L}^2. \tag{28}$$

The operator $\boldsymbol{\sigma}$ should be determined such that it satisfies Eq. (28). In order to be able to compare the both sides of the equation, we will represent $\boldsymbol{L}$ by position $\boldsymbol{x}$ and momentum $\boldsymbol{p}$. First, we expand the right-hand side of Eq. (28) as

$$\boldsymbol{L}^2 = \frac{1}{2}\sum_{i,j=1}^{n}(x_i p_j - x_j p_i)(x_i p_j - x_j p_i) = \sum_{i,j=1}^{n}(x_i p_j x_i p_j - x_i p_j x_j p_i). \tag{29}$$

Then we use the canonical commutation relation (9) so as to move all x_k to the left of p_k, as follows:

$$\boldsymbol{L}^2 = \sum_{i,j=1}^{n} (x_i x_i p_j p_j - x_i x_j p_i p_j) + i\hbar(n-1)\sum_{i=1}^{n} x_i p_i. \tag{30}$$

In the same manner, the left-hand side of Eq. (28) is expanded as

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{L})^2 + \hbar(n-2)\,\boldsymbol{\sigma}\cdot\boldsymbol{L} &= \frac{1}{4}\sum_{i,j,k,\ell=1}^{n} \sigma_{ij}\sigma_{k\ell} L_{ij} L_{k\ell} + \frac{\hbar(n-2)}{2}\sum_{i,j=1}^{n}\sigma_{ij} L_{ij} \\ &= \sum_{i,j,k,\ell=1}^{n} \sigma_{ij}\sigma_{k\ell} x_i p_j x_k p_\ell + \hbar(n-2)\sum_{i,j=1}^{n}\sigma_{ij} x_i p_j \\ &= \sum_{i,j,k,\ell=1}^{n} \sigma_{ij}\sigma_{k\ell} x_i x_k p_j p_\ell + i\hbar\sum_{i,j,k=1}^{n}\sigma_{ik}\sigma_{jk} x_i p_j + \hbar(n-2)\sum_{i,j=1}^{n}\sigma_{ij} x_i p_j. \end{aligned}\tag{31}$$

Now, we divide the first summation about k,ℓ into two parts. One contains the cases $(k,\ell) = (i,j),(j,i)$ and the other does not contain them:

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\boldsymbol{L})^2 + \hbar(n-2)\,\boldsymbol{\sigma}\cdot\boldsymbol{L} = \sum_{i,j=1}^{n}\sigma_{ij}^2 (x_i x_i p_j p_j - x_i x_j p_i p_j) + \sum_{i,j=1}^{n}\sum_{\substack{\{k,\ell\}\\ \neq\{i,j\}}} \sigma_{ij}\sigma_{k\ell} x_i x_k p_j p_\ell \\ + i\hbar\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk} - i(n-2)\sigma_{ij}\right) x_i p_j. \end{aligned}\tag{32}$$

Comparing Eq. (32) with Eq. (30), we obtain the two identities:

$$\sum_{i,j=1}^{n}\sigma_{ij}^2 (x_i x_i p_j p_j - x_i x_j p_i p_j) + \sum_{i,j=1}^{n}\sum_{\substack{\{k,\ell\}\\ \neq\{i,j\}}} \sigma_{ij}\sigma_{k\ell} x_i x_k p_j p_\ell = \sum_{i,j=1}^{n} (x_i x_i p_j p_j - x_i x_j p_i p_j), \tag{33}$$

$$\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk} - i(n-2)\sigma_{ij}\right) x_i p_j = (n-1)\sum_{i=1}^{n} x_i p_i. \tag{34}$$

Equation (33) provides the two conditions:

$$\sigma_{ij}^2 = 1 \qquad \text{for } i \neq j, \tag{35}$$

$$\sum_{i,j=1}^{n}\sum_{\substack{\{k,\ell\}\\ \neq\{i,j\}}} \sigma_{ij}\sigma_{k\ell} x_i x_k p_j p_\ell = 0. \tag{36}$$

For Eq. (34), we divide the summation about j into two parts, $j = i$ and $j \neq i$:

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\sigma_{ik}^2 x_i p_i + \sum_{i\neq j}\left(\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk} - i(n-2)\sigma_{ij}\right) x_i p_j = (n-1)\sum_{i=1}^{n} x_i p_i. \tag{37}$$

Then, using Eq. (35), we obtain

$$\sum_{i\neq j}\left(\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk} - i(n-2)\sigma_{ij}\right) x_i p_j = 0. \tag{38}$$

This equation holds only when each coefficient of all $x_i p_j$ is equal to zero, that is,

$$\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk} = i(n-2)\sigma_{ij} \qquad \text{for } i \neq j. \tag{39}$$

The same is true of Eq. (36), namely, all the coefficients of $x_i x_k p_j p_\ell$ must be zero. However in this case, we must take care the commutativity of x_i and x_k, or p_j and p_ℓ. The coefficient of $x_i x_k p_j p_\ell$ is not $\sigma_{ij}\sigma_{k\ell}$ only but $(\sigma_{ij}\sigma_{k\ell} + \sigma_{kj}\sigma_{i\ell} + \sigma_{i\ell}\sigma_{kj} + \sigma_{k\ell}\sigma_{ij})$. With attention to the summation indices range, Eq. (36) provides the following condition:

$$\sigma_{ij}\sigma_{k\ell} + \sigma_{kj}\sigma_{i\ell} + \sigma_{i\ell}\sigma_{kj} + \sigma_{k\ell}\sigma_{ij} = 0 \qquad \text{for three or four different index values.} \tag{40}$$

Summarizing all the above, we have obtained the following properties of $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^\dagger, \tag{41}$$

$$[\sigma_{ij}, L_{k\ell}] = 0, \qquad i, j, k, \ell = 1, \ldots, n, \tag{42}$$

$$\sigma_{ij} = -\sigma_{ji}, \qquad i, j = 1, \ldots, n, \tag{43}$$

$$\sigma_{ij}^2 = 1, \qquad \text{for } i \neq j, \tag{44}$$

$$\sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} = i(n-2)\sigma_{ij}, \qquad \text{for } i \neq j, \tag{45}$$

$$\sigma_{ij}\sigma_{k\ell} + \sigma_{kj}\sigma_{i\ell} + \sigma_{i\ell}\sigma_{kj} + \sigma_{k\ell}\sigma_{ij} = 0, \qquad \text{for three or four different index values.} \tag{46}$$

These are the most general properties of $\boldsymbol{\sigma}$, though Eqs. (45) and (46) are somewhat complicated. We will simplify these two conditions in the next section.

III. ALGEBRAIC STRUCTURE

Equation (45) contains the summation symbol $\sum$. This summation is troublesome in high-dimensional space, though it disappears in low-dimensional space. Therefore, we examine the algebraic structure of $\boldsymbol{\sigma}$ especially in low-dimensional space. As a matter of fact, this low-dimensional simplicity can be inherited to higher-dimensional space if we keep the low-dimensional expressions unchanged. In this section, we will seek a simpler alternative to Eq. (45) by mathematical induction.

In the case of $n = 2$ In two-dimensional space, σ_{12} is the only significant component of $\boldsymbol{\sigma}$. The remaining components are not independent: $\sigma_{11} = \sigma_{22} = 0$, $\sigma_{21} = -\sigma_{12}$. Among the above equations (41)–(46), non-trivial conditions are the first four, and in particular, the condition (44) is important:

$$\sigma_{12}^2 = 1. \tag{47}$$

As a concrete σ_{12}, we can take the identity operator for example, which is Hermitian, commutes with $\boldsymbol{L}$ and satisfies Eq. (44).

Rather, we had better not introduce $\boldsymbol{\sigma}$ in two-dimensional space, where L_{12} is the only independent component of $\boldsymbol{L}$, and $\boldsymbol{L}^2$ is simply written as L_{12}^2. The reason is made clear by observing two-dimensional subspace of three-dimensional space. In ordinary three-dimensional quantum mechanics, we usually write L_{12} as L_z and its eigenvalue as $\hbar m$ with $m = 0, \pm1, \pm2, \ldots$, and we do not see the square operator L_z^2 and the square eigenvalue $\hbar^2 m^2$ ($\hbar^2 m^2$ corresponds to eigenvalue (13)). Not to mention, we never introduce $\boldsymbol{\sigma}$ to distinguish between non-negative m and negative $-m$ with the restriction $m \geq 0$. The same goes for pure two-dimensional space, and there is no need to use the square operator $\boldsymbol{L}^2 = L_{12}^2$ or to introduce $\boldsymbol{\sigma}$. We have only to use L_{12} from the beginning.

In the case of $n = 3$ This case corresponds to ordinary quantum mechanics. The operator $\boldsymbol{\sigma}$ has three independent components:

$$\sigma_{12},\ \sigma_{13},\ \sigma_{23}. \tag{48}$$

Equation (45) provides

$$\begin{aligned} \sigma_{13}\sigma_{23} &= i\sigma_{12}, & \sigma_{23}\sigma_{13} &= i\sigma_{21}, \\ \sigma_{12}\sigma_{32} &= i\sigma_{13}, & \sigma_{32}\sigma_{12} &= i\sigma_{31}, \\ \sigma_{21}\sigma_{31} &= i\sigma_{23}, & \sigma_{31}\sigma_{21} &= i\sigma_{32}, \end{aligned} \tag{49}$$

and these are put together as

$$\sigma_{ik}\sigma_{jk} = i\sigma_{ij} \qquad \text{for three different index values.} \tag{50}$$

From this relation and the antisymmetricity (43), we find

$$\sigma_{ik}\sigma_{jk} = i\sigma_{ij} = -i\sigma_{ji} = -\sigma_{jk}\sigma_{ik} \tag{51}$$

and obtain the anticommutativity

$$\sigma_{ik}\sigma_{jk} = -\sigma_{jk}\sigma_{ik} \qquad \text{for three different index values.} \tag{52}$$

With regard to Eq. (46), it also leads the anticommutativity (52) as proved in the following. In three-dimensional space, there are not more than three different index values, and it reduces Eq. (46) to the following non-trivial two cases. One is $\ell = k$

$$\sigma_{ij}\sigma_{kk} + \sigma_{kj}\sigma_{ik} + \sigma_{ik}\sigma_{kj} + \sigma_{kk}\sigma_{ij} = 0 \qquad \text{for three different index values,} \tag{53}$$

and the other is $\ell = j$

$$\sigma_{ij}\sigma_{kj} + \sigma_{kj}\sigma_{ij} + \sigma_{ij}\sigma_{kj} + \sigma_{kj}\sigma_{ij} = 0 \qquad \text{for three different index values.} \tag{54}$$

It is obvious that both cases yield Eq. (52), and we know that the complicated conditions (45) and (46) are reduced to the simple condition (50).

The operator $\boldsymbol{\sigma}$ can be represented by Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ if we set

$$\sigma_{12} = \sigma_z, \qquad \sigma_{13} = -\sigma_y, \qquad \sigma_{23} = \sigma_x \tag{55}$$

with

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{56}$$

In the case of $n = 4$ In four-dimensional space, the operator $\boldsymbol{\sigma}$ has the six independent components:

$$\sigma_{12},\ \sigma_{13},\ \sigma_{14},\ \sigma_{23},\ \sigma_{24},\ \sigma_{34}. \tag{57}$$

Equation (45) provides

$$\begin{aligned}
\sigma_{13}\sigma_{23} + \sigma_{14}\sigma_{24} &= 2i\sigma_{12}, & \sigma_{23}\sigma_{13} + \sigma_{24}\sigma_{14} &= 2i\sigma_{21}, \\
\sigma_{12}\sigma_{32} + \sigma_{14}\sigma_{34} &= 2i\sigma_{13}, & \sigma_{32}\sigma_{12} + \sigma_{34}\sigma_{14} &= 2i\sigma_{31}, \\
\sigma_{12}\sigma_{42} + \sigma_{13}\sigma_{43} &= 2i\sigma_{14}, & \sigma_{42}\sigma_{12} + \sigma_{43}\sigma_{13} &= 2i\sigma_{41}, \\
\sigma_{21}\sigma_{31} + \sigma_{24}\sigma_{34} &= 2i\sigma_{23}, & \sigma_{31}\sigma_{21} + \sigma_{34}\sigma_{24} &= 2i\sigma_{32}, \\
\sigma_{21}\sigma_{41} + \sigma_{23}\sigma_{43} &= 2i\sigma_{24}, & \sigma_{41}\sigma_{21} + \sigma_{43}\sigma_{23} &= 2i\sigma_{42}, \\
\sigma_{31}\sigma_{41} + \sigma_{32}\sigma_{42} &= 2i\sigma_{34}, & \sigma_{41}\sigma_{31} + \sigma_{42}\sigma_{32} &= 2i\sigma_{43}.
\end{aligned} \tag{58}$$

These relations are somewhat complicated, but if we assume that Eq. (49) holds in four-dimensional space without change, some of the relations in Eq. (58) reduce to

$$\begin{aligned}
\sigma_{14}\sigma_{24} &= i\sigma_{12}, & \sigma_{24}\sigma_{14} &= i\sigma_{21}, \\
\sigma_{14}\sigma_{34} &= i\sigma_{13}, & \sigma_{34}\sigma_{14} &= i\sigma_{31}, \\
\sigma_{24}\sigma_{34} &= i\sigma_{23}, & \sigma_{34}\sigma_{24} &= i\sigma_{32}.
\end{aligned} \tag{59}$$

Then, these six equations can be expressed as

$$\sigma_{i4}\sigma_{j4} = i\sigma_{ij} \qquad \text{for } i,j \leq 3,\ i \neq j, \tag{60}$$

and multiplying Eq. (60) by σ_{i4} from the left or by σ_{j4} from the right yields

$$\sigma_{4i}\sigma_{ji} = i\sigma_{4j}, \qquad \sigma_{ij}\sigma_{4j} = i\sigma_{i4} \qquad \text{for } i,j \leq 3,\ i \neq j. \tag{61}$$

Combining Eq. (60) and Eq. (61), we obtain the general expression:

$$\sigma_{ik}\sigma_{jk} = i\sigma_{ij} \qquad \text{for three different index values.} \tag{62}$$

Like the case of $n = 3$, Eq. (62) provides the anticommutativity:

$$\sigma_{ik}\sigma_{jk} = -\sigma_{jk}\sigma_{ik} \qquad \text{for three different index values.} \tag{63}$$

Let us see the last condition (46). As with the case of $n = 3$, Eq. (46) provides the anticommutativity (63) when two of the four indices have the same value. Therefore, it is only necessary to consider the case where all the indices i, j, k, ℓ have different values. When all the indices are different, Eq. (62) yields the anticommutativity on indices as follows:

$$\sigma_{ij}\sigma_{k\ell} = -i\sigma_{i\ell}\sigma_{j\ell}\sigma_{k\ell} = i\sigma_{i\ell}\sigma_{k\ell}\sigma_{j\ell} = -\sigma_{i\ell}\sigma_{kj}. \tag{64}$$

In a similar way, it follows that

$$\sigma_{kj}\sigma_{i\ell} = -\sigma_{k\ell}\sigma_{ij}, \tag{65}$$

and adding Eq. (64) and Eq. (65), we obtain

$$\sigma_{ij}\sigma_{k\ell} + \sigma_{kj}\sigma_{i\ell} + \sigma_{i\ell}\sigma_{kj} + \sigma_{k\ell}\sigma_{ij} = 0. \tag{66}$$

Therefore, we do not have to consider the complicated condition (46) because it can be deduced from the more fundamental relation (62). By the same calculation as Eq. (64), we can show the following commutativity:

$$\sigma_{ij}\sigma_{k\ell} = \sigma_{k\ell}\sigma_{ij} \qquad \text{for four different index values.} \tag{67}$$

This commutativity is important in considering matrix representation of $\boldsymbol{\sigma}$.

In the case of $n \geq 5$ Consider the case of $n = N$ with $N \geq 5$. Here, suppose that we have already obtained the following equation in $(N-1)$-dimensional space:

$$\sigma_{ik}\sigma_{jk} = i\sigma_{ij} \qquad \text{for three different index values.} \tag{68}$$

Now, we inductively prove that Eq. (68) also holds in N-dimensional space if the relation $\sigma_{ik}\sigma_{jk} = i\sigma_{ij}$ is assumed as correct for the index subset $i, j, k \leq N-1$. In the case of $i, j \neq N$, the left-hand side of Eq. (45) can be calculated as

$$\sum_{k=1}^{N}\sigma_{ik}\sigma_{jk} = \left(\sum_{k=1}^{N-1}\sigma_{ik}\sigma_{jk}\right) + \sigma_{iN}\sigma_{jN} = i(N-3)\sigma_{ij} + \sigma_{iN}\sigma_{jN} \qquad \text{for } i, j \leq N-1,\ i \neq j, \tag{69}$$

and the whole of Eq. (45) becomes

$$\sigma_{iN}\sigma_{jN} = i\sigma_{ij} \qquad \text{for } i, j \leq N-1,\ i \neq j. \tag{70}$$

Then, multiplication of this equation by σ_{iN} from the left or by σ_{jN} from the right yields

$$\sigma_{Ni}\sigma_{ji} = i\sigma_{Nj}, \qquad \sigma_{ij}\sigma_{Nj} = i\sigma_{iN} \qquad \text{for } i, j \leq N-1,\ i \neq j. \tag{71}$$

Combining Eq. (70) and Eq. (71), we know that Eq. (68) also holds in N-dimensional space, and by mathematical induction, it holds for all $n \geq 3$. Note that the anticommutativity $\sigma_{ik}\sigma_{jk} = -\sigma_{jk}\sigma_{ik}$ is automatically satisfied when Eq. (68) holds, and it leads Eq. (46).

For arbitrary dimensions, the components of the Hermitian operator $\boldsymbol{\sigma}$ has the following algebraic structure:

$$\sigma_{ij} = -\sigma_{ji}, \tag{72}$$

$$\sigma_{ij}^2 = 1, \qquad \text{for } i \neq j, \tag{73}$$

$$\sigma_{ik}\sigma_{jk} = i\sigma_{ij}, \qquad \text{for three different index values,} \tag{74}$$

$$\sigma_{ik}\sigma_{jk} = -\sigma_{jk}\sigma_{ik}, \qquad \text{for three different index values,} \tag{75}$$

$$\sigma_{ij}\sigma_{k\ell} = \sigma_{k\ell}\sigma_{ij}, \qquad \text{for four different index values.} \tag{76}$$

Equations (75) and (76) are derivative of Eq. (74), but we leave them written down for convenience. From Eqs. (72)–(76), we can make sure that the following commutation relation holds:

$$[\,\sigma_{ij}, \sigma_{k\ell}\,] = 2i(\delta_{ik}\sigma_{j\ell} + \delta_{i\ell}\sigma_{kj} + \delta_{jk}\sigma_{\ell i} + \delta_{j\ell}\sigma_{ik}). \tag{77}$$

If we define an operator $\boldsymbol{S}$ as

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}, \tag{78}$$

it satisfies the angular momentum commutation relation:

$$[\, S_{ij}, S_{k\ell}\,] = i\hbar(\delta_{ik}S_{j\ell} + \delta_{i\ell}S_{kj} + \delta_{jk}S_{\ell i} + \delta_{j\ell}S_{ik}). \tag{79}$$

In the case of $n = 3$, the operator $\boldsymbol{S}$ is identified with spin of ordinary quantum mechanics. Note that $\boldsymbol{S}$ does not satisfy a relation similar to Eq. (12) when all the four indices i, j, k, ℓ have different values.

IV. MATRIX REPRESENTATION

We represent the components of $\boldsymbol{\sigma}$ by matrices in such a way as to hold the algebraic structure (72)–(76).

General properties We will see the general properties of a matrix σ_{ij} before studying matrix representation concretely. First and most obviously, σ_{ij} is a Hermitian matrix. In addition, σ_{ij} must be a unitary matrix because the matrix version of Eq. (73),

$$\sigma_{ij}^2 = I \tag{80}$$

gives $\sigma_{ij}^{-1} = \sigma_{ij} = \sigma_{ij}^{\dagger}$. Note that the imaginary unit i in Eq. (74) prevents all components being real matrices for $n \geq 3$.

Hermitian matrices can always be diagonalized by proper unitary matrices. Here, we use the notation U for a unitary matrix which diagonalizes σ_{ij}. Multiplication of Eq. (80) by U, U^{-1} yields

$$(U^{-1}\sigma_{ij}U)(U^{-1}\sigma_{ij}U) = I. \tag{81}$$

This equation indicates that the eigenvalues of σ_{ij} are 1 or -1 only. For $n \geq 3$, making use of Eqs. (73) and (75) yields

$$\sigma_{ij} = \sigma_{ij}\sigma_{jk}\sigma_{jk} = -\sigma_{jk}\sigma_{ij}\sigma_{jk}. \tag{82}$$

With the trace property $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$, the trace of Eq. (82) is given by

$$\mathrm{tr}\,\sigma_{ij} = -\mathrm{tr}\,\sigma_{ij}, \tag{83}$$

and we obtain

$$\mathrm{tr}\,\sigma_{ij} = 0 \qquad \text{for } n \geq 3. \tag{84}$$

Note that this traceless property prevents us from using $\pm I$ as components of $\boldsymbol{\sigma}$. The traceless property also holds for diagonalized σ_{ij}:

$$\mathrm{tr}\,(U^{-1}\sigma_{ij}U) = 0 \qquad \text{for } n \geq 3. \tag{85}$$

This implies that the number of eigenvalue 1 is equal to that of -1, and the order of matrices must be an even number for $n \geq 3$. When the order of $\sigma_{ij}$is given by $2k$, the determinant is calculated as

$$\det \sigma_{ij} = \det(U^{-1}\sigma_{ij}U) = (-1)^k. \tag{86}$$

Here, we prove that a representation matrix of a certain component σ_{ij} must be different from representation matrices of all the other components of $\boldsymbol{\sigma}$. For $n \geq 3$, there exist two independent components σ_{ij} and $\sigma_{k\ell}$ ($\{i, j\} \neq \{k, \ell\}$), and temporarily suppose that both are represented by the same matrix. This assumption implies that the matrices σ_{ij} and $\sigma_{k\ell}$ are commutative and it means that all the four indices i, j, k, ℓ must be different from one another because of Eqs. (75) and (76). In three-dimensional space, where we can never select four different index values, the necessity of four different values is apparent contradiction and it suggests that the initial assumption is wrong. In the case of $n \geq 5$, we can always pick out four different indices i, j, k, ℓ without problems, and in addition, one more different index m. With these five different indices, we can image three independent components $\sigma_{ij}, \sigma_{k\ell}, \sigma_{\ell m}$.

From Eqs. (75) and (76), the three components must satisfy the following commutation relation and anticommutation relation:

$$\sigma_{ij}\sigma_{\ell m} = \sigma_{\ell m}\sigma_{ij}, \qquad \sigma_{k\ell}\sigma_{\ell m} = -\sigma_{\ell m}\sigma_{k\ell}. \tag{87}$$

Now, remember $\sigma_{ij} = \sigma_{k\ell}$ as matrices, and then Eq. (87) becomes

$$\sigma_{ij}\sigma_{\ell m} = \sigma_{\ell m}\sigma_{ij}, \qquad \sigma_{ij}\sigma_{\ell m} = -\sigma_{\ell m}\sigma_{ij}, \tag{88}$$

and these yield

$$\sigma_{ij}\sigma_{\ell m} = 0. \tag{89}$$

The multiplication of Eq. (89) by $\sigma_{\ell m}$ from the right gives

$$\sigma_{ij} = 0. \tag{90}$$

However, the zero matrix cannot satisfy Eq. (73), and it indicates that the initial assumption $\sigma_{ij} = \sigma_{k\ell}$ is wrong. Thus all the representation matrices must be different from one another except the case $n = 4$.

To summarize the above, a representation matrix σ_{ij} has the following properties:

- Both Hermitian and unitary matrix: $\sigma_{ij} = \sigma_{ij}^{\dagger} = \sigma_{ij}^{-1}$;
- For $n \geq 3$, σ_{ij} is an even order matrix and is not the identity matrix;
- For $n \geq 3$, when the order of σ_{ij} is $2k$, the trace and determinant are given by $\mathrm{tr}\,(\sigma_{ij}) = 0$, $\det\,(\sigma_{ij}) = (-1)^k$;
- Except $n = 4$, the representation matrix of σ_{ij} is different from those of all the other components.

In the following, we will concretely seek representation matrices of σ_{ij} in low-dimensional spaces ($n = 3, 4, 5, 6$). We are particularly interested in constant representation, which does not depend on any parameters. Once a certain representation is found, then other representations will be generated by unitary transformations of the representation because the algebraic structure (72)–(76) is preserved under unitary transformations.

In the case of $n = 3$ In three-dimensional space, the possible orders of representation matrices are $2, 4, 6, \ldots$. For simplicity, we seek 2×2 matrix representation first. Larger size matrices are examined only when suitable representations cannot be found within 2×2 matrices.

In general, a traceless 2×2 Hermitian matrix has the following form:

$$\begin{bmatrix} p & q + ir \\ q - ir & -p \end{bmatrix}, \tag{91}$$

where p, q, r are real numbers. The unitarity and the determinant -1 property both yield

$$p^2 + q^2 + r^2 = 1. \tag{92}$$

The three independent components $\sigma_{12}, \sigma_{13}, \sigma_{23}$ must satisfy Eq. (74) and Eq. (92) simultaneously. As a trial, we fix the form of the primary component σ_{12} to be diagonal:

$$\sigma_{12} = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{93}$$

where σ_z is the Pauli matrices notation (56). The remaining two components σ_{13}, σ_{23} must be determined so that both anticommute with σ_{12}. The anticommutator between σ_{12} and the matrix (91) is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p & q + ir \\ q - ir & -p \end{bmatrix} + \begin{bmatrix} p & q + ir \\ q - ir & -p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2p & 0 \\ 0 & 2p \end{bmatrix}. \tag{94}$$

This equation tells that σ_{13} and σ_{23} cannot have the diagonal entries ($p = 0$). Taking account of the correspondence to the Pauli matrices, we decide to choose the matrix σ_{23} as

$$\sigma_{23} = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{95}$$

From the relation $\sigma_{13} = i\sigma_{12}\sigma_{23}$, the form of σ_{13} is automatically settled as

$$\sigma_{13} = -\sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \tag{96}$$

We can make sure that the set of the three matrices $\sigma_{12}, \sigma_{13}, \sigma_{23}$ satisfies Eq. (74) properly.

In the case of $n = 4$ In four-dimensional space, $\boldsymbol{\sigma}$ has six independent components. Like the three-dimensional space, we try to seek representation within the limits of 2×2 matrices.

First, for the subset $\sigma_{12}, \sigma_{13}, \sigma_{23}$, we elect to use the same representation as $n = 3$:

$$\sigma_{12} = \sigma_z, \quad \sigma_{13} = -\sigma_y, \quad \sigma_{23} = \sigma_x. \tag{97}$$

The remaining components $\sigma_{14}, \sigma_{24}, \sigma_{34}$ must have the matrix form (91) also. From the condition (76), we can see that σ_{34} commutes with σ_{12}, and so consider the commutator between σ_{12} and the matrix (91):

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p & q+ir \\ q-ir & -p \end{bmatrix} - \begin{bmatrix} p & q+ir \\ q-ir & -p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2(q+ir) \\ -2(q-ir) & 0 \end{bmatrix}. \tag{98}$$

This indicates that the non-diagonal entries of σ_{34} must be zero, and the form is restricted to

$$\begin{bmatrix} p & 0 \\ 0 & -p \end{bmatrix}, \qquad p = \pm 1. \tag{99}$$

The case $p = 1$ leads the matrix equality $\sigma_{34} = \sigma_{12}$, and the case $p = -1$ leads $\sigma_{34} = \sigma_{21}$, but four-dimensional space exceptionally allows to assign one specific matrix to two or more components of $\boldsymbol{\sigma}$. Here we choose the case $p = 1$:

$$\sigma_{34} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{100}$$

The remaining components σ_{14} and σ_{24} are automatically determined as follows:

$$\sigma_{14} = i\sigma_{13}\sigma_{34} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{101}$$

$$\sigma_{24} = i\sigma_{23}\sigma_{34} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{102}$$

Direct calculation shows that the obtained matrices fulfill all the conditions (72)–(76).

In the case of $n = 5, 6$ In five-dimensional space, the four new components $\sigma_{15}, \sigma_{25}, \sigma_{35}, \sigma_{45}$ are added, and the total number of independent components becomes ten. First of all, we must extend $n = 4$ representation

$$\begin{aligned} \sigma_{12} &= \sigma_z, & \sigma_{13} &= -\sigma_y, & \sigma_{23} &= \sigma_x, \\ \sigma_{14} &= \sigma_x, & \sigma_{24} &= \sigma_y, & \sigma_{34} &= \sigma_z \end{aligned} \tag{103}$$

to suitable form, because in five-dimensional or higher-dimensional space, it is forbidden to assign a certain matrix to two or more components of $\boldsymbol{\sigma}$. To this end, we raise the size of the matrices to 4×4.

The tensor product is a simple way to extend 2×2 matrices to 4×4. Generally, the tensor product of a $p \times q$ matrix $A = [a_{ij}]$ and an $r \times s$ matrix $B = [b_{ij}]$ is the $pr \times qs$ matrix with the following form:

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{bmatrix}. \tag{104}$$

Note that block matrix notation is used in this equation. The multiplication of two tensor products is calculated as

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \tag{105}$$

The Hermitian conjugate of tensor product is

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger. \tag{106}$$

Using the tensor product, we simply extend the subset $\sigma_{12}, \sigma_{13}, \sigma_{23}$ to 4×4 matrices as follows:

$$\sigma_{12} = \sigma_z \otimes I, \qquad \sigma_{13} = -\sigma_y \otimes I, \qquad \sigma_{23} = \sigma_x \otimes I, \tag{107}$$

where I is the identity matrix of size 2. For $\sigma_{14}, \sigma_{24}, \sigma_{34}$, we choose the following representation so that six matrices are all different from one another:

$$\sigma_{14} = \sigma_x \otimes \sigma_z, \qquad \sigma_{24} = \sigma_y \otimes \sigma_z, \qquad \sigma_{34} = \sigma_z \otimes \sigma_z. \tag{108}$$

We can make sure that matrices (107), (108) satisfy the $n = 4$ algebra properly.

Let us determine the remaining four components $\sigma_{15}, \sigma_{25}, \sigma_{35}, \sigma_{45}$ within the tensor product form of two Pauli matrices. To begin with, note that σ_{45} is commutative with the matrices (107) and anticommutative with (108). It restricts the possible forms of σ_{45} to $\pm(I \otimes \sigma_x)$ or $\pm(I \otimes \sigma_y)$. Here we choose the following form in an arbitrary manner:

$$\sigma_{45} = I \otimes \sigma_x. \tag{109}$$

With Eq. (105), the remaining components are automatically determined from the relation $\sigma_{i5} = i\sigma_{i4}\sigma_{45}$:

$$\sigma_{15} = -\sigma_x \otimes \sigma_y, \qquad \sigma_{25} = -\sigma_y \otimes \sigma_y, \qquad \sigma_{35} = -\sigma_z \otimes \sigma_y. \tag{110}$$

In the same way as $n = 5$, we can find $n = 6$ matrices easily. If the above ten matrices are adopted without change, all we have to do is to set the remaining five components $\sigma_{16}, \sigma_{26}, \sigma_{36}, \sigma_{46}, \sigma_{56}$ properly. Note that the last component σ_{56} commutes with the six matrices (107), (108). This commutativity restricts the possible forms of σ_{56} to $\pm(I \otimes \sigma_z)$, and here we choose

$$\sigma_{56} = I \otimes \sigma_z. \tag{111}$$

From the relation $\sigma_{i6} = i\sigma_{i5}\sigma_{56}$, the remaining four components are automatically determined as

$$\sigma_{16} = \sigma_x \otimes \sigma_x, \qquad \sigma_{26} = \sigma_y \otimes \sigma_x, \qquad \sigma_{36} = \sigma_z \otimes \sigma_x, \qquad \sigma_{46} = I \otimes \sigma_y. \tag{112}$$

Now we have obtained all the matrices of $n = 5, 6$ with consideration of only a part of the necessary conditions. However, by calculation, we can check that all the conditions (72)–(76) are exhaustively satisfied. The result of $n = 5, 6$ is explicitly written as follows (no entries indicates 0):

$$\sigma_{12} = \sigma_z \otimes I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}, \qquad \sigma_{13} = -\sigma_y \otimes I = \begin{bmatrix} & & i & \\ & & & i \\ -i & & & \\ & -i & & \end{bmatrix}, \qquad \sigma_{23} = \sigma_x \otimes I = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix},$$

$$\sigma_{14} = \sigma_x \otimes \sigma_z = \begin{bmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{bmatrix}, \qquad \sigma_{24} = \sigma_y \otimes \sigma_z = \begin{bmatrix} & & -i & \\ & & & i \\ i & & & \\ & -i & & \end{bmatrix}, \qquad \sigma_{34} = \sigma_z \otimes \sigma_z = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix},$$

$$\sigma_{15} = -\sigma_x \otimes \sigma_y = \begin{bmatrix} & & & i \\ & & -i & \\ & i & & \\ -i & & & \end{bmatrix}, \qquad \sigma_{25} = -\sigma_y \otimes \sigma_y = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}, \qquad \sigma_{35} = -\sigma_z \otimes \sigma_y = \begin{bmatrix} & i & & \\ -i & & & \\ & & & -i \\ & & i & \end{bmatrix},$$

$$\sigma_{16} = \sigma_x \otimes \sigma_x = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}, \qquad \sigma_{26} = \sigma_y \otimes \sigma_x = \begin{bmatrix} & & & -i \\ & & -i & \\ & i & & \\ i & & & \end{bmatrix}, \qquad \sigma_{36} = \sigma_z \otimes \sigma_x = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & -1 & \end{bmatrix},$$

$$\sigma_{45} = I \otimes \sigma_x = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \qquad \sigma_{46} = I \otimes \sigma_y = \begin{bmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{bmatrix}, \qquad \sigma_{56} = I \otimes \sigma_z = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

Representation matrices in higher-dimensional spaces are given in Appendix.

V. EXAMPLE

Let us solve the eigenvalue problem of $\boldsymbol{\sigma} \cdot \boldsymbol{L}$ in four-dimensional space ($n = 4$). Here we adopt the 2×2 matrix representation (103). The operator $\boldsymbol{\sigma} \cdot \boldsymbol{L}$ is expressed as

$$\boldsymbol{\sigma} \cdot \boldsymbol{L} = \begin{bmatrix} L_{12} + L_{34} & (L_{23} + L_{14}) - i(L_{31} + L_{24}) \\ (L_{23} + L_{14}) + i(L_{31} + L_{24}) & -(L_{12} + L_{34}) \end{bmatrix}, \tag{113}$$

or simply written as

$$\boldsymbol{\sigma} \cdot \boldsymbol{L} = \begin{bmatrix} M & M_- \\ M_+ & -M \end{bmatrix} \tag{114}$$

with

$$M = L_{12} + L_{34}, \qquad M_{\pm} = (L_{23} + L_{14}) \pm i(L_{31} + L_{24}). \tag{115}$$

The eigenvalue equation is

$$\begin{bmatrix} M & M_- \\ M_+ & -M \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}, \tag{116}$$

where λ and $\begin{bmatrix} u \\ v \end{bmatrix}$ stand for the eigenvalue and eigenvector of $\boldsymbol{\sigma} \cdot \boldsymbol{L}$ respectively. If our consideration in the previous sections is valid, we will obtain eigenvalues $\lambda = \hbar\ell, -\hbar(\ell + 2)$. We will verify it in the following.

First we study the properties of $M, M_{\pm}$. Using Eq. (10), we can make sure of the following commutation relations:

$$[M, M_{\pm}] = \pm 2\hbar M_{\pm}, \tag{117}$$

$$[M_+, M_-] = 4\hbar M. \tag{118}$$

The anticommutation relation between M_+ and M_- is given by

$$M_+M_- + M_-M_+ = 2\left((L_{23} + L_{14})^2 + (L_{31} + L_{24})^2\right). \tag{119}$$

Further, adding M^2, we notice the following equation,

$$M^2 + \frac{1}{2}(M_+M_- + M_-M_+) = \boldsymbol{L}^2 + 2(L_{12}L_{34} - L_{13}L_{24} + L_{14}L_{23}), \tag{120}$$

and making use of Eq. (12), the anticommutation relation is rewritten as

$$M_+M_- + M_-M_+ = 2\boldsymbol{L}^2 - 2M^2. \tag{121}$$

The sum and difference of Eq. (118) and Eq. (121) provide

$$M_+M_- = \boldsymbol{L}^2 - M^2 + 2\hbar M, \tag{122}$$

$$M_-M_+ = \boldsymbol{L}^2 - M^2 - 2\hbar M. \tag{123}$$

From Eq. (11), it is obvious that $\boldsymbol{L}^2$ and M are commutative, and it implies that simultaneous eigenfunctions of $\boldsymbol{L}^2$ and M exist. A simultaneous eigenfunction y satisfies the following eigenvalue equations:

$$\boldsymbol{L}^2 y = \Lambda y, \qquad My = \mu y, \tag{124}$$

where Λ and μ are eigenvalues. We have already known the eigenvalues of $\boldsymbol{L}^2$ in four-dimensional space, $\Lambda = \hbar^2\ell(\ell+2)$ with $\ell = 0, 1, 2, \ldots$, but we go forward without the knowledge here. The eigenfunction y is supposed to be normalized:

$$\int_{S^3} |y|^2 \, d\Omega = 1, \tag{125}$$

where S^3 is the surface of the 3-sphere. Note that Λ never becomes negative because of

$$\Lambda = \int_{S^3} y^* \boldsymbol{L}^2 y \, d\Omega = \int_{S^3} (\boldsymbol{L}y)^* \cdot (\boldsymbol{L}y) \, d\Omega \geq 0. \tag{126}$$

Now we will show that a function M_+y is one of the simultaneous eigenfunctions of $\boldsymbol{L}^2$ and M when y satisfies Eq. (124). From the commutation relation (11), M_+y is obviously an eigenfunction of $\boldsymbol{L}^2$ with eigenvalue Λ:

$$\boldsymbol{L}^2(M_+y) = \Lambda(M_+y). \tag{127}$$

With regard to the operator M, we obtain eigenvalue $\mu + 2\hbar$ from Eq. (117):

$$M(M_+y) = (\mu + 2\hbar)(M_+y). \tag{128}$$

Similarly, M_-y is one of the simultaneous eigenfunctions of $\boldsymbol{L}^2$ and M:

$$\boldsymbol{L}^2(M_-y) = \Lambda(M_-y), \tag{129}$$

$$M(M_-y) = (\mu - 2\hbar)(M_-y). \tag{130}$$

Therefore, once we find some simultaneous eigenfunction y, then we can immediately find a series of simultaneous eigenfunctions $M_\pm y$, $M_\pm^2 y, \dots$ with eigenvalues $\mu \pm 2\hbar, \mu \pm 4\hbar, \dots$. At a glance, there seem to exist an infinite number of simultaneous eigenfunctions for the operator M, and eigenvalue μ has no upper or lower limits. However they are not true, and eigenvalue μ is bounded as proved below. With the relation $M_+^\dagger = M_-$ and Eq. (123), the integration of $|M_+ y|^2$ is

$$\int_{S^3} |M_+ y|^2 \, d\Omega = \int_{S^3} y^* M_- M_+ y \, d\Omega = \int_{S^3} y^* (\boldsymbol{L}^2 - M^2 - 2\hbar M) \, y \, d\Omega = \Lambda - \mu^2 - 2\hbar\mu, \tag{131}$$

and we know

$$\Lambda - \mu^2 - 2\hbar\mu \geq 0. \tag{132}$$

In the same way, the integration of $|M_- y|^2$ gives

$$\Lambda - \mu^2 + 2\hbar\mu \geq 0. \tag{133}$$

Combining these two inequalities restricts the possible range of μ as

$$-\sqrt{\Lambda + \hbar^2} + \hbar \leq \mu \leq \sqrt{\Lambda + \hbar^2} - \hbar. \tag{134}$$

Note that it depends on eigenvalue Λ. To make Eq. (134) simpler, we introduce the abbreviations ℓ and m as

$$\hbar\ell = \sqrt{\Lambda + \hbar^2} - \hbar, \qquad \hbar m = \mu, \tag{135}$$

and the inequality is rewritten as

$$-\ell \leq m \leq \ell. \tag{136}$$

Note $\ell \geq 0$. The conflict between an infinite number of eigenfunctions and the boundedness of m can be eliminated as follows. Suppose eigenvalue m has the maximum value $m_{\max}$ and the minimum value $m_{\min}$, and assume that the corresponding eigenfunctions $y_{\max}$ and $y_{\min}$ satisfy the following equations:

$$M_+ y_{\max} = 0, \qquad M_- y_{\min} = 0. \tag{137}$$

When these equations hold, Eq. (128) and (130) also hold trivially with the identically zero function for the outside range of Eq. (136), and the contradiction does not occur regardless of the number of $M_\pm$ operating times.

For a certain Λ, substituting $y_{\max}$ and $\mu_{\max} = \hbar m_{\max}$ into inequality (131) yields the following equality:

$$\Lambda - \mu_{\max}^2 - 2\hbar\mu_{\max} = 0. \tag{138}$$

Using $\ell, m_{\max}$ instead of $\Lambda, \mu_{\max}$, it is rewritten as

$$(m_{\max} - \ell\,)(m_{\max} + (\ell + 2)) = 0. \tag{139}$$

Similarly for $m_{\min}$, we obtain

$$(m_{\min} + \ell\,)(m_{\min} - (\ell + 2)) = 0. \tag{140}$$

From these equations, the maximum and minimum of m are determined as

$$m_{\max} = \ell, \qquad m_{\min} = -\ell. \tag{141}$$

A series of M_+ operation on $y_{\min}$ causes $+2$ increment for eigenvalue m at each time, and after several operations, the maximum eigenvalue $m_{\max}$ and the corresponding eigenfunction $y_{\max}$ must be obtained to be bounded above. Therefore the relation between $m_{\max}$ and $m_{\min}$ is written as $m_{\max} = m_{\min} + 2k$, where $k = 0, 1, 2, \dots$ denotes the number of times of M_+ operation, and inserting Eq. (141) into the relation yields

$$\ell = k = 0, 1, 2, \dots. \tag{142}$$

Namely, ℓ is a non-negative integer and m is a two by two stepping integer which lies in $-\ell \leq m \leq \ell$.

With the use of ℓ and m, we label a simultaneous eigenfunction of $\boldsymbol{L}^2$ and M as $y_{\ell m}$:

$$\boldsymbol{L}^2 y_{\ell m} = \hbar^2 \ell(\ell+2) y_{\ell m}, \qquad M y_{\ell m} = \hbar m y_{\ell m}. \tag{143}$$

The state labeled ℓ and $m+2$ can be obtained by M_+ operation on $y_{\ell m}$, but it is not normalized as is. From Eq. (131), we can see that $M_+ y_{\ell m}$ is related with normalized $y_{\ell m+2}$ as

$$M_+ y_{\ell m} = \hbar\sqrt{(\ell-m)(\ell+m+2)}\; y_{\ell m+2}, \tag{144}$$

where the real and positive phase is chosen. Similarly, $M_- y_{\ell m}$ is related with normalized $y_{\ell m-2}$ as

$$M_- y_{\ell m} = \hbar\sqrt{(\ell+m)(\ell-m+2)}\; y_{\ell m-2}. \tag{145}$$

We are ready to solve the original eigenvalue equation (116)

$$\begin{aligned} Mu + M_- v &= \lambda u, \\ M_+ u - Mv &= \lambda v. \end{aligned} \tag{146}$$

Careful observation of Eq. (146) makes us notice the following solution:

$$u = A y_{\ell m}, \quad v = B y_{\ell m+2}, \tag{147}$$

where A and B are some proper constants. Note that this solution also becomes an eigenstate of $\boldsymbol{L}^2$ with eigenvalue $\hbar^2\ell(\ell+2)$. In order to set the values of A, B, insert the solution (147) into Eq. (146), and it follows that

$$\begin{aligned} \hbar m A + \hbar\sqrt{(\ell+m+2)(\ell-m)}\, B &= \lambda A, \\ \hbar\sqrt{(\ell-m)(\ell+m+2)}\, A - \hbar(m+2)B &= \lambda B \end{aligned} \tag{148}$$

or

$$\begin{bmatrix} \lambda - \hbar m & -\hbar\sqrt{(\ell+m+2)(\ell-m)} \\ -\hbar\sqrt{(\ell-m)(\ell+m+2)} & \lambda + \hbar(m+2) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{149}$$

For a non-trivial solution, the determinant of the matrix must become zero:

$$(\lambda - \hbar\ell)(\lambda + \hbar(\ell+2)) = 0. \tag{150}$$

Therefore, as expected, we obtain the eigenvalues of $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ as

$$\lambda = \hbar\ell,\ -\hbar(\ell+2), \qquad \ell = 0, 1, 2, \dots. \tag{151}$$

The relation between A and B are given by

$$B = \sqrt{\frac{\ell-m}{\ell+m+2}}\, A \qquad \text{for } \lambda = \hbar\ell, \tag{152}$$

$$B = -\sqrt{\frac{\ell+m+2}{\ell-m}}\, A \qquad \text{for } \lambda = -\hbar(\ell+2). \tag{153}$$

The absolute values of A, B are determined from the normalization condition

$$\int_{S^3} [\, u^* \ v^* \,] \begin{bmatrix} u \\ v \end{bmatrix} d\Omega = \int_{S^3} \left(|u|^2 + |v|^2\right) d\Omega = 1, \tag{154}$$

and they become

$$A = \sqrt{\frac{\ell+m+2}{2\ell+2}}, \qquad B = \sqrt{\frac{\ell-m}{2\ell+2}} \qquad \text{for } \lambda = \hbar\ell, \tag{155}$$

$$A = \sqrt{\frac{\ell-m}{2\ell+2}}, \qquad B = -\sqrt{\frac{\ell+m+2}{2\ell+2}} \qquad \text{for } \lambda = -\hbar(\ell+2), \tag{156}$$

where real phase is chosen. Taken together, the eigenvalues and eigenfunctions of the operator $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ are

$$\lambda = \hbar\ell, \qquad \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2\ell+2}} \begin{bmatrix} \sqrt{\ell+m+2}\; y_{\ell m} \\ \sqrt{\ell-m}\; y_{\ell m+2} \end{bmatrix}, \tag{157}$$

$$\lambda = -\hbar(\ell+2), \quad \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{2\ell+2}} \begin{bmatrix} \sqrt{\ell-m}\; y_{\ell m} \\ -\sqrt{\ell+m+2}\; y_{\ell m+2} \end{bmatrix}. \tag{158}$$

This is the four-dimensional version of spin spherical harmonics.

VI. SUMMARY

We have investigated the analog of the spin angular momentum operator in general dimensional space. When the square of the orbital angular momentum operator was decomposed into the not squared operator $\boldsymbol{\sigma}\cdot\boldsymbol{L}$, the spinlike operator $\boldsymbol{\sigma}$ emerged as the coefficient of $\boldsymbol{L}$. Every property of $\boldsymbol{\sigma}$ was derived from the following identical equation:

$$\boldsymbol{\sigma}\cdot\boldsymbol{L}\,(\boldsymbol{\sigma}\cdot\boldsymbol{L} + \hbar(n-2)) = \boldsymbol{L}^2. \tag{159}$$

This is the most important equation in this paper. The conditional equations directly obtained from the above equation have been somewhat complicated, and we transformed them into the simpler equations in such a way that relations for lower-dimensional subset were unchanged in higher-dimensional space. Then, by heuristic method, we found the concrete representation matrices in several low-dimensional spaces with attention to the restriction condition on the matrix form. Lastly, we solved the eigenvalue problem of $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ in four-dimensional space and obtained the consistent eigenvalues.

In ordinary three-dimensional quantum mechanics, spin is assumed to be independent of orbital angular momentum. However, in our derivation of spin, $\boldsymbol{\sigma}$ always accompanies $\boldsymbol{L}$, and we do not have to deal with the eigenvalues and eigenvectors of $\boldsymbol{\sigma}$ or spin $\boldsymbol{S} = \hbar\boldsymbol{\sigma}/2$ directly. Spin or $\boldsymbol{\sigma}$ is nothing but a coefficient, and $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ seems to be more fundamental existence. In a hydrogen-like atom, degenerate states of the electron are usually distinguish by both the orbital angular momentum quantum number and spin quantum number. However, in our new interpretation, the states can be distinguished by the orbital angular momentum quantum number $\hbar\ell, -\hbar(\ell+n-2)$ only. The not squared operator $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ enables us to unify spin into orbital angular momentum.

Appendix

In the case of $n \geq 7$, examples of $\boldsymbol{\sigma}$ matrices are given. The use of the Pauli matrices and tensor product makes it easy to find proper representations.

$\boldsymbol{n = 7, 8}$

$$\sigma_{12} = \sigma_z \otimes I \otimes I, \qquad \sigma_{13} = -\sigma_y \otimes I \otimes I, \qquad \sigma_{23} = \sigma_x \otimes I \otimes I$$

$$\sigma_{14} = \sigma_x \otimes \sigma_z \otimes \sigma_z, \qquad \sigma_{24} = \sigma_y \otimes \sigma_z \otimes \sigma_z, \qquad \sigma_{34} = \sigma_z \otimes \sigma_z \otimes \sigma_z$$

$$\sigma_{15} = -\sigma_x \otimes \sigma_y \otimes \sigma_z, \qquad \sigma_{25} = -\sigma_y \otimes \sigma_y \otimes \sigma_z, \qquad \sigma_{35} = -\sigma_z \otimes \sigma_y \otimes \sigma_z$$

$$\sigma_{16} = \sigma_x \otimes \sigma_x \otimes \sigma_z, \qquad \sigma_{26} = \sigma_y \otimes \sigma_x \otimes \sigma_z, \qquad \sigma_{36} = \sigma_z \otimes \sigma_x \otimes \sigma_z$$

$$\sigma_{45} = I \otimes \sigma_x \otimes I, \qquad \sigma_{46} = I \otimes \sigma_y \otimes I, \qquad \sigma_{56} = I \otimes \sigma_z \otimes I$$

$$\sigma_{17} = -\sigma_x \otimes I \otimes \sigma_y, \qquad \sigma_{27} = -\sigma_y \otimes I \otimes \sigma_y, \qquad \sigma_{37} = -\sigma_z \otimes I \otimes \sigma_y$$

$$\sigma_{47} = I \otimes \sigma_z \otimes \sigma_x, \qquad \sigma_{57} = -I \otimes \sigma_y \otimes \sigma_x, \qquad \sigma_{67} = I \otimes \sigma_x \otimes \sigma_x$$

$$\sigma_{18} = \sigma_x \otimes I \otimes \sigma_x, \qquad \sigma_{28} = \sigma_y \otimes I \otimes \sigma_x, \qquad \sigma_{38} = \sigma_z \otimes I \otimes \sigma_x$$

$$\sigma_{48} = I \otimes \sigma_z \otimes \sigma_y, \qquad \sigma_{58} = -I \otimes \sigma_y \otimes \sigma_y, \qquad \sigma_{68} = I \otimes \sigma_x \otimes \sigma_y$$

$$\sigma_{78} = I \otimes I \otimes \sigma_z$$

$n = 9, 10$

$\sigma_{12} = \sigma_z \otimes I \otimes I \otimes I,$ $\sigma_{13} = -\sigma_y \otimes I \otimes I \otimes I,$ $\sigma_{23} = \sigma_x \otimes I \otimes I \otimes I$

$\sigma_{14} = \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{24} = \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{34} = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$

$\sigma_{15} = -\sigma_x \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{25} = -\sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{35} = -\sigma_z \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z$

$\sigma_{16} = \sigma_x \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{26} = \sigma_y \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{36} = \sigma_z \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z$

$\sigma_{45} = I \otimes \sigma_x \otimes I \otimes I,$ $\sigma_{46} = I \otimes \sigma_y \otimes I \otimes I,$ $\sigma_{56} = I \otimes \sigma_z \otimes I \otimes I$

$\sigma_{17} = -\sigma_x \otimes I \otimes \sigma_y \otimes \sigma_z,$ $\sigma_{27} = -\sigma_y \otimes I \otimes \sigma_y \otimes \sigma_z,$ $\sigma_{37} = -\sigma_z \otimes I \otimes \sigma_y \otimes \sigma_z$

$\sigma_{47} = I \otimes \sigma_z \otimes \sigma_x \otimes I,$ $\sigma_{57} = -I \otimes \sigma_y \otimes \sigma_x \otimes I,$ $\sigma_{67} = I \otimes \sigma_x \otimes \sigma_x \otimes I$

$\sigma_{18} = \sigma_x \otimes I \otimes \sigma_x \otimes \sigma_z,$ $\sigma_{28} = \sigma_y \otimes I \otimes \sigma_x \otimes \sigma_z,$ $\sigma_{38} = \sigma_z \otimes I \otimes \sigma_x \otimes \sigma_z$

$\sigma_{48} = I \otimes \sigma_z \otimes \sigma_y \otimes I,$ $\sigma_{58} = -I \otimes \sigma_y \otimes \sigma_y \otimes I,$ $\sigma_{68} = I \otimes \sigma_x \otimes \sigma_y \otimes I$

$\sigma_{19} = -\sigma_x \otimes I \otimes I \otimes \sigma_y,$ $\sigma_{29} = -\sigma_y \otimes I \otimes I \otimes \sigma_y,$ $\sigma_{39} = -\sigma_z \otimes I \otimes I \otimes \sigma_y$

$\sigma_{49} = I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x,$ $\sigma_{59} = -I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_x,$ $\sigma_{69} = I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_x$

$\sigma_{1\,10} = \sigma_x \otimes I \otimes I \otimes \sigma_x,$ $\sigma_{2\,10} = \sigma_y \otimes I \otimes I \otimes \sigma_x,$ $\sigma_{3\,10} = \sigma_z \otimes I \otimes I \otimes \sigma_x$

$\sigma_{4\,10} = I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_y,$ $\sigma_{5\,10} = -I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_y,$ $\sigma_{6\,10} = I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_y$

$\sigma_{78} = I \otimes I \otimes \sigma_z \otimes I,$ $\sigma_{79} = -I \otimes I \otimes \sigma_y \otimes \sigma_x,$ $\sigma_{89} = I \otimes I \otimes \sigma_x \otimes \sigma_x$

$\sigma_{7\,10} = -I \otimes I \otimes y \otimes y,$ $\sigma_{8\,10} = I \otimes I \otimes \sigma_x \otimes \sigma_y,$ $\sigma_{9\,10} = I \otimes I \otimes I \otimes \sigma_z$

$n = 11, 12$

$\sigma_{12} = \sigma_z \otimes I \otimes I \otimes I \otimes I,$ $\sigma_{13} = -\sigma_y \otimes I \otimes I \otimes I \otimes I,$ $\sigma_{23} = \sigma_x \otimes I \otimes I \otimes I \otimes I$

$\sigma_{14} = \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{24} = \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{34} = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$

$\sigma_{15} = -\sigma_x \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{25} = -\sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{35} = -\sigma_z \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$

$\sigma_{16} = \sigma_x \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{26} = \sigma_y \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{36} = \sigma_z \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$

$\sigma_{45} = I \otimes \sigma_x \otimes I \otimes I \otimes I,$ $\sigma_{46} = I \otimes \sigma_y \otimes I \otimes I \otimes I,$ $\sigma_{56} = I \otimes \sigma_z \otimes I \otimes I \otimes I$

$\sigma_{17} = -\sigma_x \otimes I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{27} = -\sigma_y \otimes I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{37} = -\sigma_z \otimes I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z$

$\sigma_{47} = I \otimes \sigma_z \otimes \sigma_x \otimes I \otimes I,$ $\sigma_{57} = -I \otimes \sigma_y \otimes \sigma_x \otimes I \otimes I,$ $\sigma_{67} = I \otimes \sigma_x \otimes \sigma_x \otimes I \otimes I$

$\sigma_{18} = \sigma_x \otimes I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{28} = \sigma_y \otimes I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z,$ $\sigma_{38} = \sigma_z \otimes I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z$

$\sigma_{48} = I \otimes \sigma_z \otimes \sigma_y \otimes I \otimes I,$ $\sigma_{58} = -I \otimes \sigma_y \otimes \sigma_y \otimes I \otimes I,$ $\sigma_{68} = I \otimes \sigma_x \otimes \sigma_y \otimes I \otimes I$

$\sigma_{19} = -\sigma_x \otimes I \otimes I \otimes \sigma_y \otimes \sigma_z,$ $\sigma_{29} = -\sigma_y \otimes I \otimes I \otimes \sigma_y \otimes \sigma_z,$ $\sigma_{39} = -\sigma_z \otimes I \otimes I \otimes \sigma_y \otimes \sigma_z$

$\sigma_{49} = I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes I,$ $\sigma_{59} = -I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_x \otimes I,$ $\sigma_{69} = I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_x \otimes I$

$\sigma_{1\,10} = \sigma_x \otimes I \otimes I \otimes \sigma_x \otimes \sigma_z,$ $\sigma_{2\,10} = \sigma_y \otimes I \otimes I \otimes \sigma_x \otimes \sigma_z,$ $\sigma_{3\,10} = \sigma_z \otimes I \otimes I \otimes \sigma_x \otimes \sigma_z$

$\sigma_{4\,10} = I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_y \otimes I,$ $\sigma_{5\,10} = -I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_y \otimes I,$ $\sigma_{6\,10} = I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_y \otimes I$

$\sigma_{1\,11} = -\sigma_x \otimes I \otimes I \otimes I \otimes \sigma_y,$ $\sigma_{2\,11} = -\sigma_y \otimes I \otimes I \otimes I \otimes \sigma_y,$ $\sigma_{3\,11} = -\sigma_z \otimes I \otimes I \otimes I \otimes \sigma_y$

$\sigma_{4\,11} = I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x,$ $\sigma_{5\,11} = -I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x,$ $\sigma_{6\,11} = I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x$

$\sigma_{1\,12} = \sigma_x \otimes I \otimes I \otimes I \otimes \sigma_x,$ $\sigma_{2\,12} = \sigma_y \otimes I \otimes I \otimes I \otimes \sigma_x,$ $\sigma_{3\,12} = \sigma_z \otimes I \otimes I \otimes I \otimes \sigma_x$

$\sigma_{4\,12} = I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_y,$ $\sigma_{5\,12} = -I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z \otimes \sigma_y,$ $\sigma_{6\,12} = I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_y$

$\sigma_{78} = I \otimes I \otimes \sigma_z \otimes I \otimes I,$ $\sigma_{79} = -I \otimes I \otimes \sigma_y \otimes \sigma_x \otimes I,$ $\sigma_{89} = I \otimes I \otimes \sigma_x \otimes \sigma_x \otimes I$

$\sigma_{7\,10} = -I \otimes I \otimes \sigma_y \otimes \sigma_y \otimes I,$ $\sigma_{8\,10} = I \otimes I \otimes \sigma_x \otimes \sigma_y \otimes I,$ $\sigma_{9\,10} = I \otimes I \otimes I \otimes \sigma_z \otimes I$

$\sigma_{7\,11} = -I \otimes I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_x,$ $\sigma_{8\,11} = I \otimes I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_x,$ $\sigma_{9\,11} = -I \otimes I \otimes I \otimes \sigma_y \otimes \sigma_x$

$\sigma_{7\,12} = -I \otimes I \otimes \sigma_y \otimes \sigma_z \otimes \sigma_y,$ $\sigma_{8\,12} = I \otimes I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_y,$ $\sigma_{9\,12} = -I \otimes I \otimes I \otimes \sigma_y \otimes \sigma_y$

$\sigma_{10\,11} = I \otimes I \otimes I \otimes \sigma_x \otimes \sigma_x,$ $\sigma_{10\,12} = I \otimes I \otimes I \otimes \sigma_x \otimes \sigma_y,$ $\sigma_{11\,12} = I \otimes I \otimes I \otimes I \otimes \sigma_z$