Characterizing Dynamics of a Physical System

How the dynamics of a physical system is investigated

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Abstract

In this paper, we shall study an electromechanical system, which is capable of showing chaos. Our aim would be to identify deterministic chaos, which effectively means finding out the conditions in which the system would show aperiodicity. We shall study a coupled system, consisting of a Bullard Dynamo driving a Faraday Disk. First, we shall give a brief description of the system (by stating the equations describing the system), then we shall identify the fixed points for the system, and identify different dynamical regimes. Our objective here is to elaborate the methods by which a dynamical system is characterized.

1 Introduction

Any system with quantities, which vary with time, is labelled as a *Dynamical System*. Almost any system of interest is a dynamical system, because we are almost always interested in predicting the evolution of a system, rather than studying the conditions of equilibrium (not to say, that studying equilibrium conditions isn't important). Out of all sorts of systems, that are under study in the modern research, the most important ones are those of nonlinear systems. The recognition of the fact that whole is greater than the sum of its parts, lies at the crux of all the new work being done on such systems. But, what do they study?
One of the most wonderful discoveries of the later half 20th century is, 'deterministic chaos'. It has its origins in the study of nonlinear systems, and the most credited of them is Edward Lorenz's seminal paper [8]. The word 'deterministic' is because of the fact that time evolution of the system doesn't involve any random variables i.e. they are not stochastic. However, the strange (exciting) part is that some of these systems can exhibit chaotic i.e. aperiodic motion.
To study this aperiodic behavior i.e. chaos, we have chosen an electromechanical system. The particular choice of this system is motivated by the

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ubiquity of electromechanical systems in day-to-day life. For the system, we will take a Bullard Dynamo and a Faraday Disk. A Faraday disk or a homopolar disk is a conducting disk(or a cylinder) rotating in a uniform magnetic field. Owing to rotation, a potential difference is generated between the rims of the disk and the axis of rotation. A Bullard dynamo, studied in details in [2], is a self-exciting homopolar disk. The current produced by the homopolar disk is used to create the magnetic field for the disk.

2 Bullard Dynamo and Faraday Disk

The basic description of the system can be found in [6]. The fundamental idea is to use the current I produced by rotating a Bullard dynamo, which is being driven by a constant torque Γ, and hence moving with a variable angular velocity Ω_b, to drive the homopolar disk. The coefficient of mutual inductance between coil of Bullard Dynamo and its disk, is $2\pi M$ and L is the self-inductance of the coil. By passing a current through the disk, we are applying the emf generated by the Bullard dynamo, across the axis and the rim of the disk, as shown in the fig. (1). So, as a result the homopolar disk

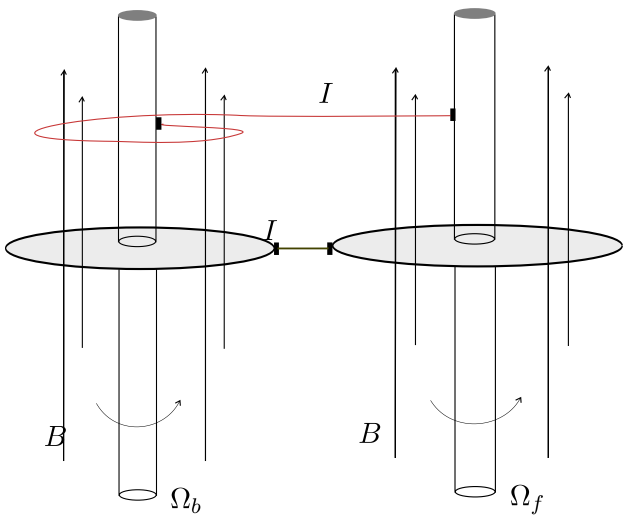

Figure 1: Bullard Dynamo driving a Faraday disk

will start rotating with an angular velocity of Ω_f. For simplicity we shall assume that the disks are identical i.e. have the same moment of inertia

J and have the same coefficient of friction λ. We shall assume that the dynamics of the system are be governed by three variables, and hence our phase space would be (I, Ω_b, Ω_f).
Consider the following things:

- The emf generated by a Faraday disk of radius a with an angular velocity Ω_f, would be $\frac{1}{2}\Omega_f a^2 B$. We shall use $\alpha := \frac{1}{2}a^2$ for convenience of notation. Hence, a Faraday disk is supplied with emf of $\alpha B \Omega_f$ it will rotate with Ω_f.
- When a Bullard dynamo has an instantaneous velocity of Ω_b and has an instantaneous current of I, it will generate an emf of $M\Omega_b I$, where $2\pi M$ is the mutual inductance between the coil and the disk.

The dynamical equations for the system are:

$$\begin{aligned}
\dot{I} &= \left(\frac{M\Omega_b - R}{L}\right) I - \frac{\alpha B}{L}\Omega_f \\
\dot{\Omega_b} &= \frac{\Gamma}{J} - \frac{\lambda}{J}\Omega_b - \frac{MI^2}{J} \\
\dot{\Omega_f} &= \frac{\alpha B}{J} I - \frac{\lambda}{J}\Omega_f
\end{aligned}$$

Before we can do any meaningful study of these, we should make these equations dimensionless. After the following choice of variables:

$$\begin{aligned}
t &= \frac{L}{R}t' \\
I &= \frac{JR^2}{\alpha B L^2}x \\
\Omega_b &= \frac{R}{M}y \\
\Omega_f &= \frac{R}{M}z
\end{aligned}$$

we get this set of equations:

$$\dot{x} = x(y-1) - \gamma z \tag{1}$$
$$\dot{y} = \kappa - \beta y - \chi x^2 \tag{2}$$
$$\dot{z} = \gamma x - \beta z \tag{3}$$

where,

$$\begin{aligned}
\beta &= \frac{\lambda L}{JR} \\
\gamma &= \frac{M}{L} \\
\kappa &= \frac{\Gamma M L}{JR^2} \\
\chi &= \frac{M^2 J^2 R^2}{\alpha^2 B^2 L^3}
\end{aligned}$$

The technique of making equations dimensionless is important when we want to discuss topics like 'long term behavior' etc. Words like 'longterm' have no meaning for dimensional equations because, if something is long for one system, might be very short for another. Take for example, the atomic system. 2 seconds might be a very long time for atomic processes, but is nothing for macroscopic mechanical systems.
Notice that, we have a 4 dimensional parameter space here. We shall have a different sorts of behavior for the system, based on the values of the parameters.

3 Fixed points of the System

For a system described like

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

the fixed points are the points in the phase space for which

$$\mathbf{f}(\mathbf{x} = 0)$$

. For our system, these points are

$$\begin{aligned}
\mathbf{F_1} &= \left(0, \frac{\kappa}{\beta}, 0\right) \\
\mathbf{F_2} &= \left(\frac{\Lambda}{\sqrt{\chi}}, \left\{1 + \frac{\gamma^2}{\beta}\right\}, \frac{\gamma\Lambda}{\beta\sqrt{\chi}}\right) \\
\mathbf{F_3} &= \left(\frac{-\Lambda}{\sqrt{\chi}}, \left\{1 + \frac{\gamma^2}{\beta}\right\}, \frac{-\gamma\Lambda}{\beta\sqrt{\chi}}\right)
\end{aligned}$$

where $\Lambda = \sqrt{\kappa - \beta - \gamma^2}$. A crucial point to be noted is out of these, $\mathbf{F_2}$ and $\mathbf{F_3}$ exist only if Λ is real i.e. if $\kappa - \beta > \gamma^2$. So, we have 4 parameters to play with, and with different settings we expect different behaviors. To know what values should we set to set different behvaiors, we should do a linear stability analysis around these fixed points.

4 Stability of Fixed Points

The Jacobian matrix for the system

$$\mathbf{Df}(x, y, z) = \begin{pmatrix} y - 1 & x & -\gamma \\ -2\chi x & -\beta & 0 \\ \gamma & 0 & -\beta \end{pmatrix} \tag{4}$$

For $\mathbf{F_1}$
Let the eigenvalues for the Jacobian matrix at $\mathbf{F_1}$ be μ_i. The characteristic equation turns out to be

$$(\mu+\beta)[\mu^2+\mu(\beta+1-\frac{\kappa}{\beta})+\beta-\kappa+\gamma^2]=0$$

So, one of the eigenvalues, $\mu_1=-\beta$ is always negative.
The signs of the other two, μ_2 and μ_3 depend on the values of β, κ and γ.

$$\mu_2+\mu_3=\frac{k}{\beta}-\beta-1 \tag{5}$$

$$\mu_2\mu_3=\beta-\kappa+\gamma^2 \tag{6}$$

If $\mu_2\mu_3<0$, $\mathbf{F_2}$ and $\mathbf{F_3}$ are created. Also, this condition will make $\mathbf{F_1}$ unstable, because one of μ_2 and μ_3 would be negative.
So, as κ passes the value of $\beta+\gamma^2$, two new fixed points are formed the other one looses stability. We have a *pitchfork bifurcation.* In the parameter space, κ and χ are the only parameters which can be influenced externally. Since, χ doesn't enter into the discussion of the stability of the fixed points, κ is the bifurcation parameter here.
If $\kappa<\beta+\gamma^2$, we have only $\mathbf{F_1}$. For stability,

$$\beta^2+\beta-\kappa>0$$

When this is true, we have the flow simply ending up at $\mathbf{F_1}$. When this is not satisfied, numerical simulations show that the flow settles into an attractor, shown in figure. (2). This is not a transient attractor. For different initial conditions too, the orbit settles into the same attractor. We shall study the exact nature of the attractor in the next chapter.
For Fixed point $\mathbf{F_2}$ and $\mathbf{F_3}$
$\mathbf{F_2}$ and $\mathbf{F_3}$ have the same characteristic equation,

$$(\mu+\beta)[\mu^2+\mu\left(\beta-\frac{\gamma^2}{\beta}\right)+2\Lambda^2]$$

One of the eigenvalues, like F_1 will be $\mu_1=-\beta$. As for μ_2 and μ_3 we have,

$$\mu_2+\mu_3 = \frac{\gamma^2}{\beta}-\beta \tag{7}$$

$$\mu_2\mu_3 = 2\Lambda^2 \tag{8}$$

Since, $2\Lambda^2>0$, μ_2 and μ_3 will have same signs. However, we can see rich behavior, because, there are chances of complex eigenvalues, mixed with real ones. The way to play with these values is to try and make all the three

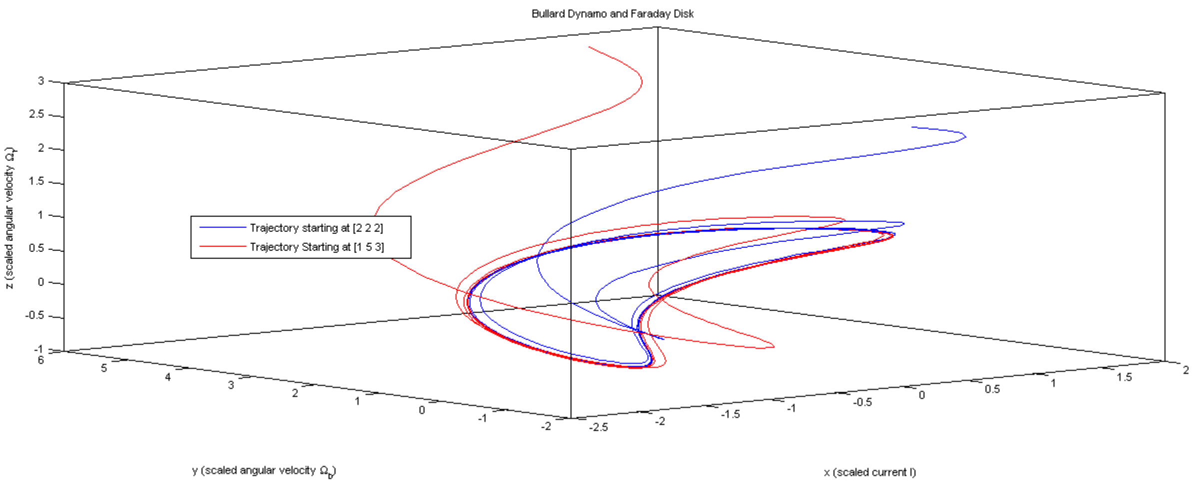

Figure 2: Trajectories starting at [2,2,2] and [1,5,3], with $\beta = 0.8$, $\gamma = 1.3$, $\kappa = 4$, $\chi = 4$, time it is run for is 200 units

fixed points unstable. We have already seen what happens when we have $\beta^2 + \beta - \kappa < 0$. For a particular choice, of parameters,

$$\beta = 2, \ \gamma = 2, \ \kappa = 20, \ \chi = 4$$

we have a very interesting sort of orbit. It is shown in fig. (3). Also, the same settings are allowed to run for longer i.e. 3000 units of time, the image is particularly heavy [fig. (4)] and seems to have filled the region in question. It should be noted that this is not a transient attractor. Orbits settle into the same attractor for different initial conditions. Different parameter

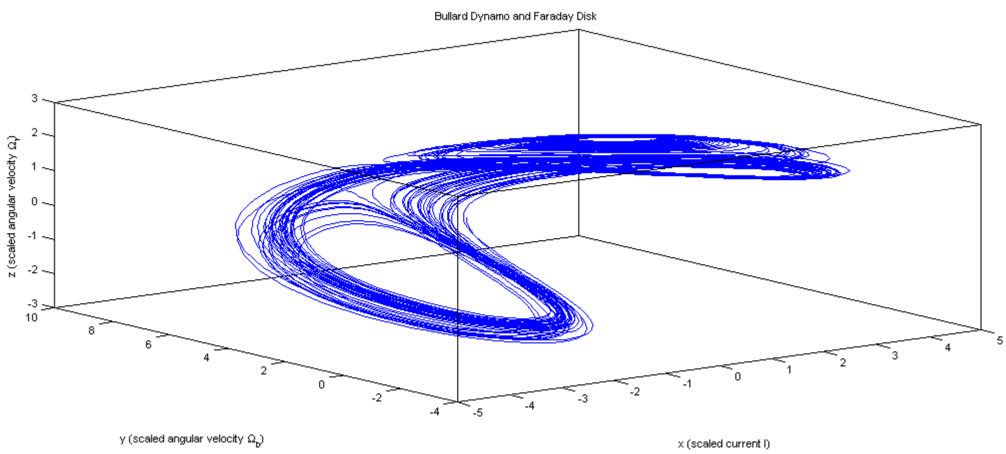

Figure 3: Trajectory, starting at [2,2,2], with $\beta = 2$, $\gamma = 2$, $\kappa = 20$, $\chi = 4$, time it is run for is 200 units of time

settings yield all sorts of behaviors. Most of the settings would make up for a situation in which the values of current and the angular velocities settle into one of the fixed values. It may spiral into that value, or it may be attracted like a node.
Now that we have established two different scenarios, our next task would be to characterize these. We shall illustrate different tools and methods that can be used to draw conclusions.

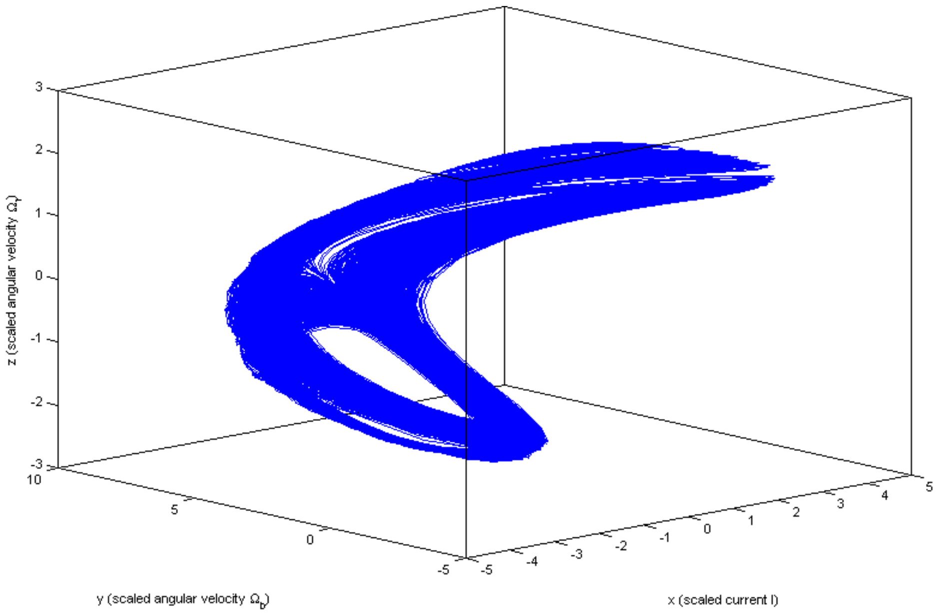

Figure 4: Trajectory, starting at [2,2,2], with $\beta = 2,\ \gamma = 2,\ \kappa = 20,\ \chi = 4$, time it is run for is 3000 units of time

5 Characterizing the Dynamics

We define two scenarios from the beginning to avoid any redundancies. These two scenarios are basically two set of values for the parameters.
Scenario 1
Will be referred to as $\mathbf{S_1}$ has the following parameter values:

$$\beta = 0.8,\ \gamma = 1.3,\ \kappa = 4,\ \chi = 4$$

Orbits starting at any point, for $\mathbf{S_1}$, eventually end up in an attractor shown in fig.(2).
Scenario 2
Will be referred to as $\mathbf{S_2}$ and has the parameter values as:

$$\beta = 2,\ \gamma = 2,\ \kappa = 20,\ \chi = 4$$

Orbits starting at any point, for $\mathbf{S_2}$, eventually end up in an attractor shown in fig.(3).

5.1 Lyapunov Exponents

While Chaos is a well established phenomenon, the characterization of chaos is still under scrutiny. Lyapunov exponents have proved to be the most useful diagnostic tool for dynamical system. Hence, we start by finding the spectrum of Lyapunov exponents for the system.
The Lyapunov exponent is defined as

$$\begin{aligned}\lambda(x_0) &= \lim_{N\to\infty}\lim_{\epsilon\to 0}\frac{1}{N}\log\frac{f^N(x_0+\epsilon)-f^N(x_0)}{\epsilon} \\ &= \lim_{N\to\infty}\frac{1}{N}\log\frac{df^N(x_0)}{dx_0}\end{aligned}$$

In order words, two trajectories in phase space with a separation of $|\delta\mathbf{x}_0|$ at time $t=0$, will have a separation of $|\delta\mathbf{x}_t|$ at $t=t$, given by

$$|\delta\mathbf{x}_t| = e^{\lambda t}|\delta\mathbf{x}_0|$$

Since, the rate of divergence of trajectories can be different in different directions in the phase space, there are as many Lyapunov exponents as is the dimensionality of phase space. Lyapunov exponents basically quantify how sensitive the system is to initial conditions. They measure the growth rates of generic perturbations. The interest is usually on the maximal Lyapunov exponent (MLE). If the phase space is compact, a positive MLE would mean that the prediction of the system is impossible. The system is then labelled to be sensitively dependent on initial conditions.
A few properties of Lyapunov Exponents are as follows [9]:

1. The Lyapunov exponents are dynamical invariants i.e. do not depend metric used or the choice of the variables for the system. So, they can characterize dynamics of a system.

2. A positive MLE implies chaos only when the phase space is bounded.

3. The sum of all Lyapunov exponents give a measure of the contraction of volume in whole of phase space. For a conservative system, the sum of the Lyapunov exponents is zero, where as for a dissipative systems it is negative.

4. For a bounded system, if the flow doesn't end at a point, then at least one of the Lyapunov exponents is zero.

5. Pesin's formula relates the Lyapunov exponents to the Kolmogorov-Sinai Entropy. The sum of the positive Lyapunov exponents gives an upper bound for KS entropy i.e.

$$\mathbf{H}_{KS} < \sum_{j:\lambda_j>0} \lambda_j$$

6. The multiplicative inverse of the MLE is called the Lyapunov time. It defines, the characteristic e-folding time i.e. the time taken for the trajectory to diverge by a factor of e. It gives a time for which predictions hold value. For chaotic orbits it is finite, and for periodic orbit, expectedly, it is infinite.

Lyapunov exponents are computed by exploiting the natural tendency of **n**-dimensional volume to align along the **n** most expanding subspaces. To find the **n** largest Lyapunov exponents, one has to find the rate of expansion of an **n**-dimensional volume. The crux of the problem is to keep track of the evolution of **n** perturbations. These should be linearly independent, but the Lyapunov vectors have a nasty habit of aligning along the same direction. This leads to numerical overflow, because of the exponentially diverging solutions. To get around this, one has to use Gram-Schmidt normalization to change the base. It can be done after each step of integration, or can be done after a chosen period. A method for finding the Lyapunov exponents (all of them) was given by Benettin et.al. in [1]. We have implemented the same algorithm, with variations hinted in [10].
The code is run for two scenarios. Here are the results:

For $\mathbf{S_1}$

The time steps of Gram-Schmidt normalization was chosen to be .01, and the time interval for computation was between (0,200), starting at the initial point of (2,2,2). The spectrum converged to (with 2-digit accuracy):

$$\{0.00, -0.35, -0.52\}$$

A few comments are in order here:

- One of the Lyapunov exponents is 0. It is a corroborating result to the fact that the trajectories ultimately don't settle to a point. They end up on an attractor, as shown in the fig.(2).
- The MLE here is 0. So, we are certain that the attractor is not chaotic.
- The sum of the positive Lyapunov constants is 0. So, from Pesin's formula, $\mathbf{H}_{KS} = 0$. The motion on the attractor is hence periodic.

The attractor in $\mathbf{S_1}$ is an isolated periodic orbit i.e. a **Limit cycle**. It is a stable limit cycle, because the orbits from initial points both inside and outside of the orbit settle into it.

For $\mathbf{S_2}$

The time steps of Gram-Schmidt normalization was chosen to be .1, and the time interval for computation was between (0,10000), starting at the initial point of (2,2,2). The spectrum converged to (with 2-digit accuracy):

$$\{0.34, 0.00, -1.65\}$$

As for this scenario,

- One of the Lyapunov exponents is 0. Same as $\mathbf{S_1}$ and also matches with the fact that the orbit doesn't converge to a point but to a subspace of the phase space.
- The MLE is 0.34 i.e. positive. So, the attractor in fig.(3) is indeed a chaotic attractor.
- The Lyapunov time for the system is 2.94 units of time (the scaled time). For times much longer than 3 units of time, system is irregular.
- We have an upper bound for $\mathbf{H}_{KS}$. $\mathbf{H}_{KS} \leq 0.34$.

We can find the Kaplan-Yorke Dimension of the attractor. It is defined as [5]

$$D_L = k + \frac{\Lambda_k}{|\lambda_{k+1}|}$$

where, $\Lambda_k = \sum_{i=0}^{k} \lambda_i$, k is the largest integer such that $\sum_{i=0}^{k} \lambda_i > 0$. Also, the $\{\lambda_i\}$ should be arranged in the descending order.
For our spectrum, we have $k = 2$ and hence,

$$D_L = 2 + \frac{0.34}{1.65} = 2.21$$

So, the KY dimension for the attractor in $\mathbf{S_2}$ is 2.21.
In a way, the Kaplan-Yorke dimension can be defined as the dimension within which a cluster of initial conditions neither expand nor contract during the evolution of the system. The system of ODEs under study is not pathological, hence we can use the result proved by Ledrappier in [7], and say that this is also equal to the Information Dimension of the attractor.
Information dimension of an attractor gives a measure of the average information needed to identify an occupied box of size p. It is defined as:

$$D_1 = \lim_{\epsilon \to 0} \frac{- < \log p_\epsilon >}{\log \frac{1}{\epsilon}}$$

where $<>$ is for average over the partitions of phase space.
So, we have the information dimension equal to 2.21.

Poincaré Surface of Section

Poincaré Surface of section is a useful tool when it comes to studying the nature of dynamics of the system. The primary reason for its usefulness is that you can study a 3 dimensional system in a plane. We shall plot the Poincaré maps for two scenarios. The code used to do this is given in the last chapter.

For $\mathbf{S_1}$

Poincaré section for a trajectory with initial points as (2,2,2), with the surface $z = 0$ is shown in fig. (5). Since the points on the Poincaré map are attracted to a point, we conclude that the 3D orbit is in fact a **limit cycle**. It is an attractive limit cycle, since orbits from all the initial points ultimately settle into that orbit. This is corroborates with what was concluded with Lyapunov exponents for $\mathbf{S_1}$

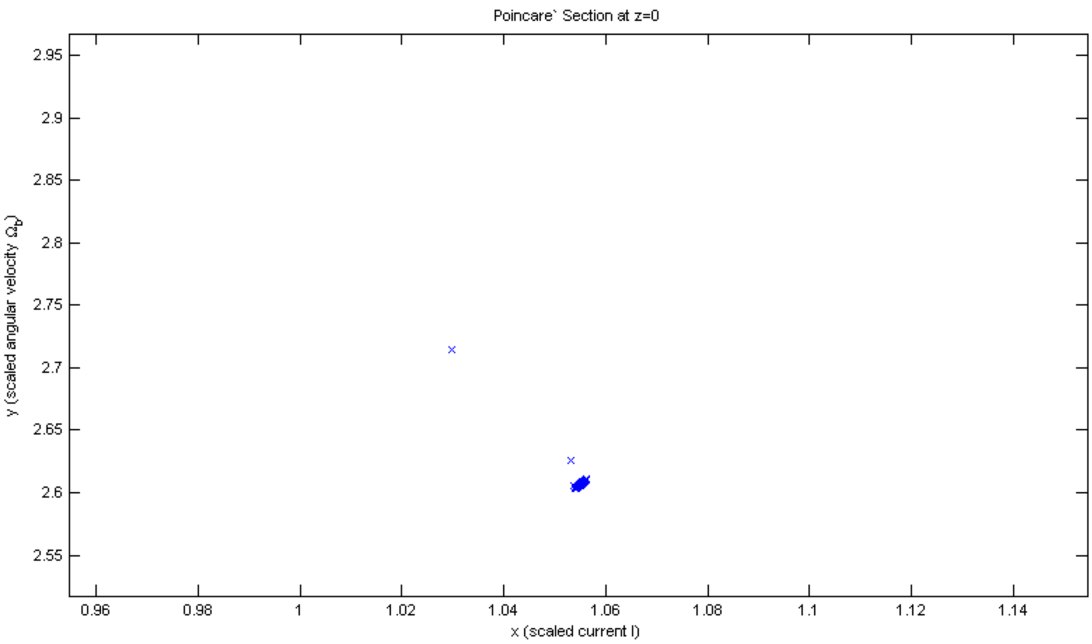

Figure 5: Trajectory, starting at [2,2,2], in setting $\mathbf{S_1}$. Time it is run for is 200 units

For $\mathbf{S_2}$

The orbit into which the trajectories settle has different sorts of structures at different parts. We take the section at $y = 2$. If each of the intersection is observed sequentially, it seems that the points jump from one region to another. To see an emerging pattern, the Poincaré sections are shown for two different runtimes. Fig. (6) shows when the simulation is run for 200 units of time. While, fig. (8) shows the Poincaré section when the

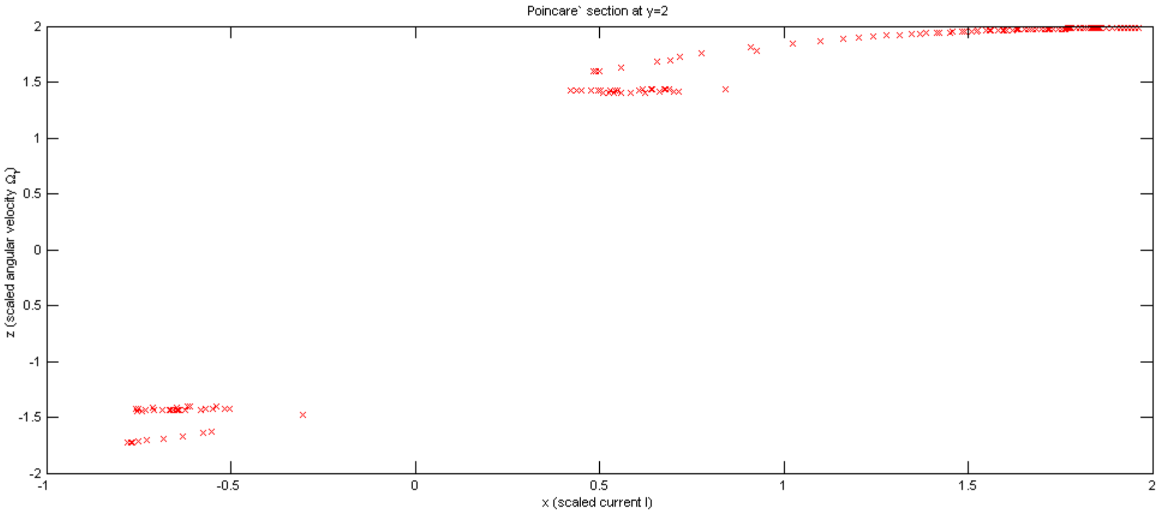

Figure 6: Trajectory, starting at [2,2,2], in setting $\mathbf{S_2}$. Time it is run for is 200 units. Section is taken at $y = 2$

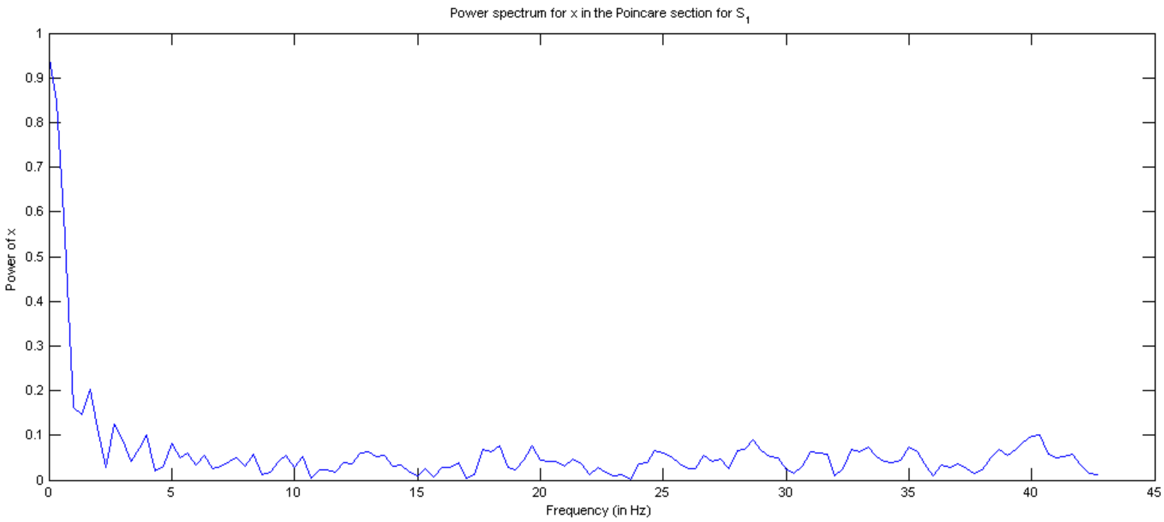

Figure 7: Power Spectrum for x for the Poincaré section, for S_1 with section $y = 2$

equations are iterated for 30000 units of time. 375001 points are generated. The Poincaré section gets 20291 intersections. Notice how densely the two regions have been filled. It starts getting filled in the first run of 200 units of time, however after 30000 units, it is obvious. This sort of space-filling of points in the Poincaré map is also a signature of Chaos.

We could have concluded from the Poincaré section from 200 units of time,

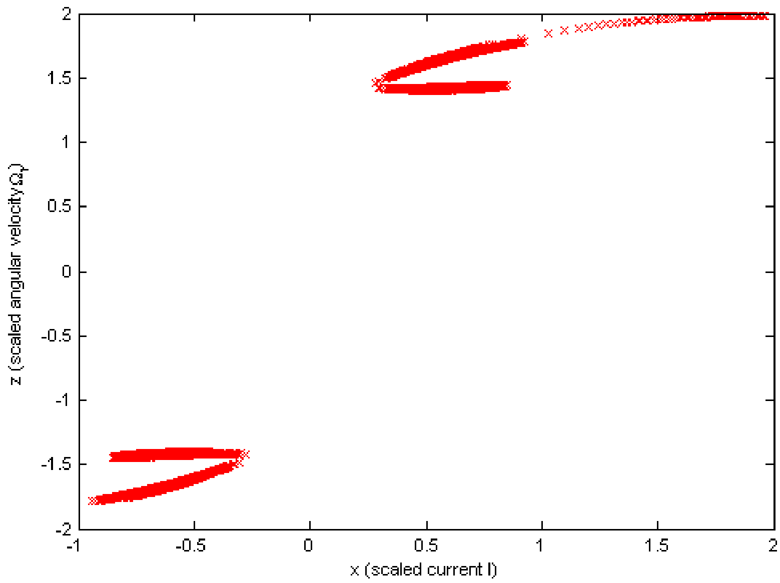

Figure 8: Trajectory, starting at [2,2,2], in setting $\mathbf{S_2}$. Time it is run for is 30000 units. Section is taken at $y = 2$

from the power spectrum fig. (7). The spectrum is continuous and hence we can conclude that there's no correlation between them.

Power Spectra

Power spectrum of a time series gives information about the periodicity of the concerned signal. The analysis of chaotic dynamics in the Fourier space reveals important information, and is especially useful for identifying transitions between periodic and chaotic regimes [3]. Power Spectra are thus very useful in discerning whether there is an order in a system or not. There are three possibilities with power spectra

1. Delta peaks at certain frequencies and integral multiple of that frequency. This happens for periodic time series.

2. Several peaks at frequencies which aren't the multiples of a particular frequency and are of different heights. This happens for quasiperiodic time series.

3. Broadband nature for the spectrum. This happens for stochastic time series.

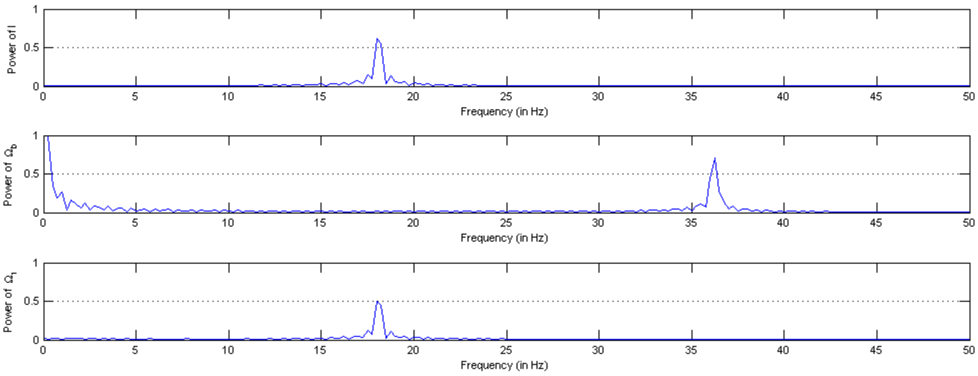

Figure 9: Power spectrum for x, y and z. Trajectory, starting at [2,2,2], in $\mathbf{S_1}$. Time it is run for is 200 units

We use the built-in Fast Fourier transform function of the MATLAB to find the power spectra of for two scenarios. The power spectra for $\mathbf{S_1}$ are shown in fig. (9). It was established that we have a limit cycle in this scenario and hence, the sharp peaks at certain frequencies, not surprisingly, indicate periodic motion.

For $\mathbf{S_2}$, the power spectra are shown in fig. (10). Clearly, except for power

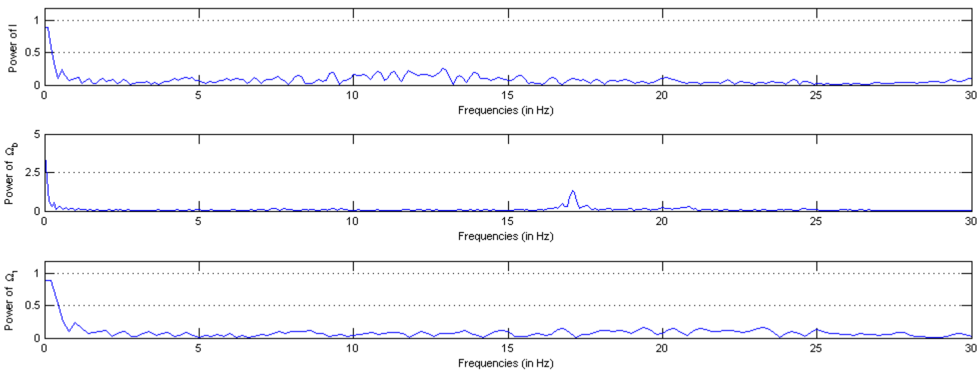

Figure 10: Power spectrum for x, y and z. Trajectory, starting at [2,2,2], in $\mathbf{S_2}$. Time it is run for is 200 units

spectrum for y, both of the spectra are broadband. Since, the dynamics were deterministic, we have deterministic chaos here.

6 Conclusion

The purpose of this work was illustrate the use certain techniques to characterize the dynamics of a physical system. You might have noticed that, other than the derivation of the dynamical equations, the fact that we are using an electromechanical system hardly came into picture. Thus, the techniques shown here are fairly general. The power of mathematical modelling is in full display here. Once the differential equations are set up, all that separated the different behaviors of the system were the value of the parameters. The parameters thus determine what sort of system we are dealing with. The system under consideration is certainly not a new one. But, by looking at it from all the facets explains how meticulously we can characterize systems.

Acknowledgement

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References

[1] G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn. Lyapunov characteristic exponents for smooth dynamical systems and for hamiltonian systems: A method for computing all of them. *Meccanica*, 15(9-30), 1980.

[2] E.C Bullard. The stability of homopolar dynamo. In *Proc. Camb. Philos. Soc.*, volume 51, page 744, 1955.

[3] J.M. Gonzalez-Miranda. *Synchronization and Control of Chaos*. Imperial College Press, 2004.

[4] P. Grassberger and Itamar Procaccia. Estimation of the kolmogorov entropy from a chaotic signal. *Physical Review A*, 1983.

[5] J. Kaplan and J. Yorke. *Chaotic behavior of multidimensional difference equations*. Lecture Notes in Mathematics, Vol 730, Springer, 1979.

[6] C. Laroche, R. Labbe, F. Petrelis, and S. Fauve. Chaotic motors. *Am. J. Phys.*, 80:113, 2012.

[7] F. Ledrappier. Some relations between dimension and lyapounov exponents. *Communications in Mathematical Physics*, 81(2):229–238, 1981.

[8] E.N Lorenz. Deterministic nonperiodic flow. *J. Atmos. Sci.*, 1936.

[9] A. Politi. Lyapunov exponent. *Scholarpedia*, 8(3):2722, 2013.

[10] Alan Wolf, Jack B. Swift, Harry L. Swinney, and John A. Vastano. Determining lyapunov exponents from a time series. *Physica D: Nonlinear Phenomena*, 16(3):285 – 317, 1985.